LECTURE NOTES

2016-03-23

GaGA: amenable groups I

1. Some context.

- Banach-Tarski (1924); measure theory
- von Neumann (1929) — first definition
- Day (1949): "amenable"; functional analytic pt. of view.

2. Def. and permanence properties.

Γ = discrete, countable grp.

Def.

- A mean for Γ on $\ell^\infty(\Gamma)$ is a linear f-tional $\sigma : \ell^\infty(\Gamma) \to \mathbb{C}$ that is positive and unital. (Means are automatically bdd.; c.f. states on C^*-alg's.)
- A mean σ is left invariant if $\sigma(sf) = \sigma(f)$ $\forall s \in \Gamma$ $\forall f \in \ell^\infty(\Gamma)$. (Here $sf = f(s^{-1}\,\cdot\,)$.)

- Γ is amenable if $\exists$ a left invariant mean for Γ on $\ell^\infty(\Gamma)$.

Ex: $\mathbb{Z}$ is amenable.

For this, recall that $\ell^1(\mathbb{Z}) \longrightarrow (\ell^\infty(\mathbb{Z}))^*$ by $g \mapsto \hat{g}$, where, writing $g = \sum_{s=-\infty}^{\infty} a_s \delta_s$,

$$\hat{g}(f) = f(g) := \sum_{s=-\infty}^{\infty} a_s f(s).$$

Let

$$P(\mathbb{Z}) = \left\{ \sum_{s=-\infty}^{\infty} a_s \delta_s \ : \ a_s \geq 0 \ \forall s \text{ and } \sum_{-\infty}^{\infty} a_s = 1 \right\};$$

this is the set of all $g \in \ell^1(\mathbb{Z})$ s.t. $\hat{g}$ is a mean on $\ell^\infty(\mathbb{Z})$. Want to find an "increasingly invariant" seq. in $P(\mathbb{Z})$ and use the fact that the unit ball in $(\ell^\infty(\mathbb{Z}))^*$ is compact in the weak-$*$ topology (Alaoglu). A cluster pt. of the seq. will be our left-inv. mean.

Just take
$$g_n = \frac{1}{2n+1} \sum_{s=-n}^{n} \delta_s .$$

Then $|\hat{g}_n(sf) - g_n(f)| \le \frac{1}{2n+1} \cdot 2s \cdot \|f\|_\infty$.

Thus any cluster pt. of (g_n) does the job. $\square$

<u>Proposition.</u>

(i) $|\Gamma| < \infty \Rightarrow \Gamma$ is amenable;

(ii) Γ = amenable $\Rightarrow$ every quotient and subgrp. of Γ is amenable;

(iii) $N \triangleleft \Gamma$ and both N and Γ/N are amenable $\Rightarrow$
$\Rightarrow$ Γ is amenable;

(iv) Γ, Λ = amenable $\Rightarrow$ $\Gamma \times \Lambda$ = amenable;

(v) an increasing union of amenable grps. is amenable;

(vi) Γ is locally amenable $\Rightarrow$ Γ is amenable

(every f.g. subgrp. ...)

(To be proved.)

<u>Corollary</u>. Γ = abelian $\Rightarrow$ Γ = amenable

(<u>N.B.</u> We're claiming this for Γ = discrete + countable, but it's true in general.)

<u>Def</u>: The class of elementary amenable grps. is the smallest class containing ~~finite & abelian grps.~~ the finite grps and the abelian grps. that is closed under taking subgrps., quotients, extensions, and direct limits.

<u>Theorem</u> (Grigorchuk) There are amenable grps. that are not elementary amenable.