ALGEBRAIC CONSISTENT QUANTUM THEORY

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1. Introduction

Let A be a bounded, self-adjoint operator on a Hilbert space $\mathcal{H}$, representing a physical quantity. Let Σ be the spectrum of A. Let

$$E : \mathcal{B}(\Sigma) \to \mathcal{P}(\mathcal{H})$$

be the spectral measure ($\mathcal{B}$ Borel measurable subsets, $\mathcal{P}$ projections). The map $E(X)$ is a property, $X \in \mathcal{B}(\Sigma)$. Each normal state is defined by a density operator ρ_0 on $\mathcal{H}$, which is positive, trace-class, $\|\rho_0\|_1 = 1$.

Let $\omega_{\rho_0} : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ be the function defined by

$$\omega_{\rho_0}(B) = Tr(\rho_0 B).$$

The trace-class ideal $\mathcal{I}^1(\mathcal{H}) = \{\text{trace class operators}\}$. In general, the trace-class norm $\|A\|_1$ of a trace-class operator A is equal to $Tr(|A|)$. The probability wrt ρ_0 of $E(X)$ is

$$\mathcal{P}^1_{\rho_0}[E(X)] = \omega_{\rho_0}(E(X)),$$

the probability of that the measurement of the observable A will be in X.

Let $A_1, ..., A_n$ be observables with corresponding spectra $\Sigma_1, ..., \Sigma_n$ and spectral measures $E_1, ..., E_n$ and a sequence of properties $E_1(X_1), ..., E_n(X_n)$ corresponding to times $t_1 < ... < t_n$. The finite sequence $(E_1(X_1), ..., E_n(X_n))$ is called a **history**. Think of this is an apparatus that makes measurements. Another proposal is that we should think of a history as a product

$$\eqsim = E_1(X_1) \odot ... \odot E_n(X_n),$$

where $\odot$ has the algebraic properties of the tensor product. We simplify $F_j = E_j(X_j)$. In the physics literature, the nth order probability of the history $\eqsim$ is defined to be

$$\mathcal{P}^n_{\rho_0}(\eqsim) = Tr(F_n F_{n-1} ... F_1 \rho_0 F_1 ... F_{n-1}).$$

For example, if

$$\eqsim' = F_1 \odot F_2,$$

then

$$\mathcal{P}^2_{\rho_0}(\eqsim') = Tr(F_2 F_1 \rho_0 F_1),$$

so then

$$\begin{aligned}\rho_0(X_1) &= \frac{F_1\rho_0F_1}{\|F_1\rho_0F_1\|}\\ &= \frac{F_1\rho_0F_1}{Tr(F_1\rho_0F_1)}\\ &= \frac{F_1\rho_0F_1}{Tr(\rho_0F_1)}\\ &= \frac{F_1\rho_0F_1}{\mathcal{P}^1_{\rho_0}(F_1)}\end{aligned}$$

is a density operator. Then

$$\begin{aligned}\mathcal{P}^2_{\rho_0}(\eqsim') &= Tr(F_2\rho_0(X_1))\,\mathcal{P}^1_{\rho_0}(F_1)\\ &= \mathcal{P}^1_{\rho_0(X_1)}(F_2)\,\mathcal{P}^1_{\rho_0}(F_1),\end{aligned}$$

the first factor of which is like a conditional probability.

Let

$$\eqsim' = F_1 \odot F_2$$

Think of $X_1 \times X_2$ as a product, with $X_1 = X_1^1 \cup X_1^2$. Then

$$\begin{aligned}E_1(X_1) &= E_1\left(X_1^1\right) \vee E_1\left(X_1^2\right)\\ &= F_1^1 \vee F_1^2,\end{aligned}$$

so that

$$\begin{aligned}\eqsim' &= \left(F_1^1 \odot F_2\right) \vee \left(F_1^2 \odot F_2\right)\\ &= \eqsim_1 \vee \eqsim_2\end{aligned}$$

Is it true that

$$\mathcal{P}^2_{\rho_0}(\eqsim') = \mathcal{P}^2_{\rho_0}(\eqsim_1 \vee \eqsim_2) = \mathcal{P}^2_{\rho_0}(\eqsim_1) + \mathcal{P}^2_{\rho_0}(\eqsim_2)?$$

Well, not generally, because

$$\begin{aligned}\mathcal{P}^2_{\rho_0}(\eqsim') &= Tr\left(F_2\left(F_1^1 \vee F_1^2\right)\rho_0\left(F_1^1 \vee F_1^2\right)\right)\\ &= Tr\left(F_2F_1^1\rho_0F_1^1\right) + Tr\left(F_2F_1^2\rho_0F_1^2\right)\\ &= \mathcal{P}^2_{\rho_0}(\eqsim_1) + \mathcal{P}^2_{\rho_0}(\eqsim_2)\\ &\quad + Tr\left(F_2F_1^1\rho_0F_1^2\right) + Tr\left(F_2F_1^2\rho_0F_1^1\right).\end{aligned}$$

The **consistency condition** for second order histories is that

$$Tr\left(F_2F_1^1\rho_0F_1^2\right) + Tr\left(F_2F_1^2\rho_0F_1^1\right) = 0$$

for every measurable decomposition of X_1 as $X_1^1 \cup X_1^2$.

We will consider $A_1, ..., A_n$ observables, $\mathcal{A}_1, ..., \mathcal{A}_n$ maximal abelian von Neumann algebras. Each of these has Gel'fand spectrum $S_1, ..., S_n$ and is endowed with a unique spectral integral

$$P_i : \boldsymbol{B}(S_i) \to \mathcal{A}_i$$

and spectral measure

$$E_i : \mathcal{B}(S_i) \to \mathcal{P}(\mathcal{A}_i).$$

Then

$$\begin{aligned} \mathbb{A}_1^n &:= \mathcal{A}_1 \overline{\otimes} ... \overline{\otimes} \mathcal{A}_n \\ \mathbb{S}_1^n &:= S_1 \times ... \times S_n \\ \mathbb{P}_1^n &:= P_1 \overline{\otimes} ... \overline{\otimes} P_n \\ \mathbb{E}_1^n &:= E_1 \overline{\otimes} ... \overline{\otimes} E_n \end{aligned}$$

Then

$$\begin{aligned} E_1^n (\mathbb{Y}_1^n) &= E_1 (Y_1) \odot ... \odot E_n (Y_n) \\ &= F_1 \odot ... \odot F_n \\ &= \mathbb{F}_1^n \end{aligned}$$

Then

$$\begin{aligned} \mathcal{P}_{\rho_0}^n [\mathbb{F}_1^n] &= Tr (F_n F_{n-1} ... F_1 \rho_0 F_1 ... F_{n-1}) \\ &= \omega_{\rho_0} (F_1) Tr (F_n F_{n-1} ... F_2 \rho_0 (Y_1) F_2 ... F_{n-1}) \\ &= \omega_{\rho_0} (F_1) \omega_{\rho_0 (Y_1)} (F_2) ... \omega_{\rho_0 \left(\mathbb{Y}_1^{n-1}\right)} (F_n) \\ &= \boldsymbol{\omega}_{\rho_0}^n \left(\mathbb{Y}_1^{n-1}\right) [\mathbb{E}_1^n (\mathbb{Y}_1^n)] = \left(\boldsymbol{\omega}_{\rho_0}^n \left(\mathbb{Y}_1^{n-1}\right) \circ \mathbb{E}_1^n\right) [\mathbb{Y}_1^n] \\ &= \boldsymbol{\mu}_{\rho_0}^n \left[\mathbb{Y}_1^{n-1}\right] (\mathbb{Y}_1^n) . \end{aligned}$$

Notation and Summary of Results for Lecture 2

(1) Elementary history: $\boldsymbol{E}_1^n(\boldsymbol{Y}_1^n) = \boldsymbol{F}_1^n$

(2) kth initial density operator: $\rho_k = \rho_0(\boldsymbol{Y}_1^k) := \dfrac{E_k(Y_k)\rho_0(\boldsymbol{Y}_1^{k-1})E_k(Y_k)}{Tr\{E_k(Y_k)\rho_0(\boldsymbol{Y}_1^{k-1})\}}$

(3) History state with respect to ρ_0 generated by $\boldsymbol{Y}_1^{n-1}$:

$$\begin{aligned} \boldsymbol{\omega}_{\rho_0}^n(\boldsymbol{Y}_1^{n-1}) &= \boldsymbol{\omega}_{\rho_0}^k(\boldsymbol{Y}_1^{k-1}) \overline{\otimes} \omega_{\rho_k} \overline{\otimes} \omega_{\rho_k(Y_{k+1})} \overline{\otimes} \cdots \overline{\otimes} \omega_{\rho_k(\boldsymbol{Y}_{k+1}^{n-1})} \\ &= \boldsymbol{\omega}_{\rho_0}^k(\boldsymbol{Y}_1^{k-1}) \overline{\otimes} \boldsymbol{\omega}_{\rho_k}^{n-k}(\boldsymbol{Y}_{k+1}^{n-1}) \end{aligned}$$

(4) History measure with respect to ρ_0 generated by $\boldsymbol{Y}_1^{n-1}$:

$$\begin{aligned} \boldsymbol{\mu}_{\rho_0}^n[\boldsymbol{Y}_1^{n-1}] &= \boldsymbol{\mu}_{\rho_0}^k[\boldsymbol{Y}_1^{k-1}] \overline{\otimes} \mu_{\rho_k} \overline{\otimes} \mu_{\rho_k(Y_{k+1})} \overline{\otimes} \cdots \overline{\otimes} \mu_{\rho_k(\boldsymbol{Y}_{k+1}^{n-1})} \\ &= \boldsymbol{\mu}_{\rho_0}^k \left[\boldsymbol{Y}_1^{k-1}\right] \overline{\otimes} \boldsymbol{\mu}_{\rho_k}^{n-k} \left[\boldsymbol{Y}_{k+1}^{n-1}\right] \end{aligned}$$

Theorem 3.1. *The nth order probability of $\boldsymbol{E}_1^n(\boldsymbol{Y}_1^n)$ with respect to ρ_0 is given by*

$$p_{\rho_0}^n[\boldsymbol{F}_1^n] = \boldsymbol{\omega}_{\rho_0}^n(\boldsymbol{Y}_1^{n-1})[\boldsymbol{F}_1^n] = \boldsymbol{\mu}_{\rho_0}^n[\boldsymbol{Y}_1^{n-1}](\boldsymbol{Y}_1^n).$$

(5) Measurable decomposition: $Y_k = Y_k^1 \sqcup Y_k^2$ for any $k, 1 \leqslant k < n$.

(6) For $i = 1, 2$: $\boldsymbol{Y}_1^m([ik) := \boldsymbol{Y}_1^{k-1} \times Y_k^i \times \boldsymbol{Y}_{k+1}^m$, $\boldsymbol{F}_1^m(ik) = \boldsymbol{E}_1^m(\boldsymbol{Y}_1^m(ik)) = \boldsymbol{F}_1^{k-1} \times F_k^i \times \boldsymbol{F}_{k+1}^m$, for $k < m \leqslant n$.

$$\rho_k^i := \frac{E_k(Y_k^i)\rho_0(\boldsymbol{Y}_1^{k-1})E_k(Y_k^i)}{Tr\{E_k(Y_k^i)\rho_0(\boldsymbol{Y}_1^{k-1})\}} = \frac{F_k^i\rho_0(\boldsymbol{Y}_1^{k-1})F_k^i}{Tr\{F_k^i\rho_0(\boldsymbol{Y}_1^{k-1})\}}. \tag{7}$$

Definition. An elementary history $\boldsymbol{E}_1^n(\boldsymbol{Y}_1^n) = \boldsymbol{F}_1^n$ is **consistent with respect to a density operator** ρ_0 if the probabiltiy is additive. That is, for each binary measurable decomposition $Y_k = Y_k^1 \sqcup Y_k^2$ for any $k, 1 \leqslant k \leqslant n$,

$$\begin{aligned} p_{\rho_0}^n[\boldsymbol{F}_1^n] &= \boldsymbol{\omega}_{\rho_0}^n(\boldsymbol{Y}_1^{n-1})[\boldsymbol{F}_1^n] \\ &= \boldsymbol{\omega}_{\rho_0}^n\big(\boldsymbol{Y}_1^{n-1}1k\big)(\boldsymbol{F}_1^n 1k) + \boldsymbol{\omega}_{\rho_0}^n\big(\boldsymbol{Y}_1^{n-1}2k\big)(\boldsymbol{F}_1^n 2k) \\ &= p_{\rho_0}^n(\boldsymbol{F}_1^n 1k) + p_{\rho_0}^n(\boldsymbol{F}_1^n 2k)\,. \end{aligned}$$

$$\eta_k^i := \frac{\omega_{\rho_0(\boldsymbol{Y}_1^{k-1})}(F_k^i)}{\omega_{\rho_0(\boldsymbol{Y}_1^{k-1})}(F_k)} = \frac{\mu_{\rho_0(\boldsymbol{Y}_1^{k-1})}(Y_k^i)}{\mu_{\rho_0(\boldsymbol{Y}_1^{k-1})}(Y_k)};\;\; 0 \leqslant \eta_k^i \leqslant 1;\;\; \text{and}\;\; \eta_k^1 + \eta_k^2 = 1. \tag{8}$$

Proposition 4.8(i). $\rho_k = \eta_k^1\rho_k^1 + \eta_k^2\rho_k^2$.

Corollary 6.4. $\boldsymbol{\mu}_{\rho_0}^n\big(\boldsymbol{Y}_1^{n-1}ik\big) \ll \boldsymbol{\mu}_{\rho_0}^n(\boldsymbol{Y}_1^{n-1})$ everywhere in $\mathfrak{B}(\boldsymbol{S}_1^n)$.

It follows that $\boldsymbol{\mu}_{\rho_k^i}^{n-k}[\mathbf{Y}_{k+1}^{n-1}] \ll \boldsymbol{\mu}_{\rho_k}^{n-k}(\mathbf{Y}_{k+1}^{n-1})$, with Radon-Nikodym derivative denoted by $\boldsymbol{\delta}_{\rho_k^i}^{n-k}[\mathbf{Y}_{k+1}^{n-1}]$.

Theorem [4.1, 4.9, 6.5]. *For an elementary history $\boldsymbol{E}_1^n(\boldsymbol{Y}_1^n)$ and a density operator ρ_0 the following four conditions are equivalent:*

(i) $\boldsymbol{E}_1^n(\mathbf{Y}_1^n)$ is consistent with respect to ρ_0;

(ii) (CH1) $Tr\{G_k^* G_k[F_k^1\rho_0(\mathbf{Y}_1^{k-1})F_k^2 + F_k^2\rho_0(\mathbf{Y}_1^{k-1})F_k^1]\} = 0$, where $G_k = F_n \cdots F_{k+1}$;

(iii) (CH4s) $\boldsymbol{\omega}_{\eta_k^1\rho_k^1+\eta_k^2\rho_k^2}^{n-k}(\mathbf{Y}_{k+1}^{n-1})[\boldsymbol{F}_{k+1}^n]$
$$= \Big\{\eta_k^1\boldsymbol{\omega}_{\rho_k^1}^{n-k}(\mathbf{Y}_{k+1}^{n-1}) + \eta_k^2\boldsymbol{\omega}_{\rho_k^2}^{n-k}(\mathbf{Y}_{k+1}^{n-1})\Big\}\,[\boldsymbol{F}_{k+1}^n];$$
and,
(iv) (CH5) the Radon-Nikodym derivatives $\boldsymbol{\delta}_{\rho_k^i}^{n-k}[\mathbf{Y}_{k+1}^{n-1}]$ satisfy the relations:

$$\boldsymbol{\delta}_{\rho_k^1}^{n-k}[\mathbf{Y}_{k+1}^{n-1}] + \boldsymbol{\delta}_{\rho_k^2}^{n-k}[\mathbf{Y}_{k+1}^{n-1}] = 1$$

almost everywhere on $\mathbf{Y}_{k+1}^n$, and

$$\varphi_{Y_k^1}\,\overline{\otimes}\,\boldsymbol{\delta}_{\rho_k^2}^{n-k}[\mathbf{Y}_{k+1}^{n-1}] + \varphi_{Y_k^2}\,\overline{\otimes}\,\boldsymbol{\delta}_{\rho_k^1}^{n-k}[\mathbf{Y}_{k+1}^{n-1}] = 0$$

almost everywhere on $\mathbf{Y}_k^n$.

Note that the condition for a second order history is that

$$Tr\left\{F_2\left(F_1^1\rho_0F_1^2 + F_1^2\rho_0F_1^1\right)\right\} = 0,$$

and (CH1) is a generalization of this.
Proof:

$$\mathcal{P}_{\rho_0}^n\left[\boldsymbol{F}_1^n\right] = \boldsymbol{\omega}_{\rho_0}^{k-1}\left(\mathbf{Y}_1^{k-2}\right)\left[\mathbf{F}_1^{n-1}\right]\boldsymbol{\omega}_{\rho_{k-1}}^{n-k+1}\left(\mathbf{Y}_k^{n-2}\right)\left[\mathbf{F}_k^n\right],$$

and

$$\begin{aligned}\boldsymbol{\omega}^{n-k+1}_{\rho_{k-1}}\left(\mathbf{Y}^{n-2}_k\right)\left[\mathbf{F}^n_k\right] &= Tr\left\{G_kF_k\rho_0\left(\mathbf{Y}^{k-1}_1\right)\left[F^1_k\vee F^2_k\right]G^*_k\right\}\\ &= Tr\left\{G_kF^1_k\rho_0\left(\mathbf{Y}^{k-1}_1\right)\left[F^1_k\right]G^*_k\right\}+Tr\left\{G_kF^2_k\rho_0\left(\mathbf{Y}^{k-1}_1\right)\left[F^2_k\right]G^*_k\right\}\\ &\quad +cross\ terms\end{aligned}$$

So the cross terms have to be zero. This implies (CH1).

For $i=1,2$,

$$\begin{aligned}\eta^i_k\boldsymbol{\omega}^{n-k+1}_{\rho^1_k}(\boldsymbol{Y}^{n-1}_k\left[\begin{smallmatrix}i\\k\end{smallmatrix}\right])\left[\mathbf{F}^1_k\right] &= \frac{\omega_{\rho_0}\left(\mathbf{Y}^{k-1}_1\right)\left(F^i_k\right)}{\omega_{\rho_0}\left(\mathbf{Y}^{k-1}_1\right)\left(F_k\right)}\omega_{\rho_0\left(\mathbf{Y}^{k-1}_1\right)}\left(F_k\right)\omega^{n-k}_{\rho^i_k}\left(\mathbf{Y}^{n-1}_{k+1}\right)\left[\mathbf{F}^n_{k+1}\right]\\ &= \omega_{\rho_0}\left(\mathbf{Y}^{k-1}_1\right)\left(F^i_k\right)\omega^{n-k}_{\rho^i_k}\left(\mathbf{Y}^{n-1}_{k+1}\right)\left[\mathbf{F}^n_k\left[\begin{smallmatrix}i\\k\end{smallmatrix}\right]\right]\\ &= \omega^{n-k+1}_{\rho_{k-1}}\left(\mathbf{Y}^{n-1}_k\left[\begin{smallmatrix}i\\k\end{smallmatrix}\right]\right)\left[\mathbf{F}^n_k\left[\begin{smallmatrix}i\\k\end{smallmatrix}\right]\right]\end{aligned}$$

Then

$$\begin{aligned}\eta^1_k\boldsymbol{\omega}^{n-k+1}_{\rho^1_k}(\mathbf{Y}^{n-1}_k\left[\begin{smallmatrix}1\\k\end{smallmatrix}\right])\left[\mathbf{F}^1_k\right]+\eta^2_k\boldsymbol{\omega}^{n-k+1}_{\rho^1_k}(\mathbf{Y}^{n-1}_k\left[\begin{smallmatrix}2\\k\end{smallmatrix}\right])\left[\mathbf{F}^1_k\right] &= \omega^{n-k+1}_{\rho_{k-1}}\left(\mathbf{Y}^{n-1}_k\left[\begin{smallmatrix}1\\k\end{smallmatrix}\right]\right)\left[\mathbf{F}^n_k\left[\begin{smallmatrix}1\\k\end{smallmatrix}\right]\right]+\omega^{n-k+1}_{\rho_{k-1}}\left(\mathbf{Y}^{n-1}_k\left[\begin{smallmatrix}2\\k\end{smallmatrix}\right]\right)\\ &= \mathcal{P}^{n-k+1}_{\rho_{k-1}}\left(\mathbf{F}^n_k\left[\begin{smallmatrix}1\\k\end{smallmatrix}\right]\right)+\mathcal{P}^{n-k+1}_{\rho_{k-1}}\left(\mathbf{F}^n_k\left[\begin{smallmatrix}2\\k\end{smallmatrix}\right]\right).\end{aligned}$$

Multiplying through by $\omega^{k-1}_{\rho_0}\left(\mathbf{Y}^{k-2}_1\right)\left(\mathbf{F}^{k-1}_1\right)$ we obtain the sum of probabilities

$$\mathcal{P}^k_{\rho_0}\left[\mathbf{F}^n_1\right]=\left\{\eta^1_k\boldsymbol{\omega}^{n-k}_{\rho^1_k}(\mathbf{Y}^{n-1}_{k+1})\left[\mathbf{F}^n_1\right]+\eta^2_k\boldsymbol{\omega}^{n-k}_{\rho^2_k}(\mathbf{Y}^{n-1}_{k+1})\left[\mathbf{F}^n_1\right]\right\}.$$

Then (CH4s) follows.

Corollary 6.9. *If $\boldsymbol{E}^n_1(\boldsymbol{Y}^n_1)$ is consistent with respect to ρ_0 then*

$$\boldsymbol{\omega}^{n-k}_{\eta^1_k\rho^1_k+\eta^2_k\rho^2_k}(\boldsymbol{Y}^{n-1}_{k+1})=\eta^1_k\boldsymbol{\omega}^{n-k}_{\rho^1_k}(\boldsymbol{Y}^{n-1}_{k+1})+\eta^2_k\boldsymbol{\omega}^{n-k}_{\rho^2_k}(\boldsymbol{Y}^{n-1}_{k+1})$$

everywhere on $\boldsymbol{\mathcal{A}}^n_{k+1}$.

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