Classifying amenable operator algebras III.

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the "hyperfinite II_1 factor"

$$\cong \overline{\bigcup_{n\geq 1} M_{3^n}(\mathbb{C})}^{WOT} =: \mathcal{R}$$

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$A_\theta := C(S^1) \rtimes_\theta \mathbb{Z}$	$L^\infty(S^1, \mu_{Lebesgue}) \rtimes_\theta \mathbb{Z} \cong R$

by Connes' Thm.

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in fact, various notions of amenability are equivalent for vNas

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- Important ingredient: $\mathcal{M} \cong \mathcal{M} \otimes R$

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- Ingredient from last slide:

$$M \text{ amenable } \mathrm{II}_1 \text{ factor} \Rightarrow M \otimes R \cong M$$

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$M_{r_1}(\mathbb{C}) \oplus \cdots \oplus M_{r_k}(\mathbb{C})$

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- Elliott '76: A, B simple AF alg's.

$$A \cong B \iff (K_0(A), K_0(A)_+, [1_A]_0) \cong$$
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Classif. should be in terms of K-theory and related invariants.

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E.g. • $L^\infty(X,\mu)$: amenable, AFD

• $C(X)$: amenable, AF only if $\dim X = 0$.

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- Seems restrictive, but a surprising # of examples are of this form, e.g. A_θ is approximated by $\bigoplus M_{n_i}(C(S^1))$

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- it turns out there are very exotic simple amenable C^*-alg's — analogous to being type II and III !
- counterexamples to Elliott conjecture (early 00s)

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*: Std. hypothesis, true in all examples, conjecturally automatic.

Strategy, in brief:

(C, Gabe, Schafhauser, Tikuisis, White)

"Lift" classification in vNa setting

and prove existence & uniqueness theorems

for maps in C^*-setting.

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This is just canonical action $\mathbb{Z} \curvearrowright \mathbb{Z}_2 = \varprojlim \mathbb{Z}/2^n\mathbb{Z}$.

Here $K_0 = \mathbb{Z}[\frac{1}{2}]$, $K_1 = \mathbb{Z}$.

Ex. 3 Theorem applies to $C(X) \rtimes \Gamma$ if

- X compact metrizable and $\dim X < \infty$
- $\Gamma \curvearrowright X$ is free
- Γ is an elementary amenable group (e.g. nilpotent, solvable, linear, ...)

(Kerr - Naryshkin '21)