

Classifying amenable operator algebras III.

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← abelian case

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$C^*(\mathbb{Z}) \not\cong C^*(\mathbb{Z}^2)$	$C^*(\Gamma)$	$\text{vN}(\Gamma)$	$\text{vN}(\mathbb{Z}) \cong \text{vN}(\mathbb{Z}^2)$

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the "hyperfinite
 II_1 factor"



$$\cong \overline{\bigcup_{n \geq 1} M_{3^n}(\mathbb{C})}^{wot} =: \mathcal{R}$$

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$C(S') \otimes_\theta \mathbb{Z}$	$L^\infty(S', \mu_{\text{Lebesgue}}) \otimes_\theta \mathbb{Z}$

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$$A_\theta := C(S^1) \rtimes_\theta \mathbb{Z} \quad \bigg| \quad L^\infty(S^1, \mu_{\text{Lebesgue}}) \rtimes_\theta \mathbb{Z} \cong \mathcal{K}$$

by Connes' Thm.



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equiv. classes of projections

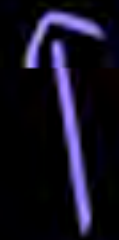
$$P \sim Q \text{ if } \begin{cases} P = v^* v \\ Q = v v^* \end{cases}$$

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projections $/_{\sim} = [0, 1]$

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in fact, various notions of amenability
are equivalent for vNa s

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- Important ingredient: $M \cong M \otimes \mathcal{K}$

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• Ingredient from last slide:

$$M \text{ amenable } \Pi_1 \text{ factor} \Rightarrow M \otimes \mathcal{R} \cong M$$

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$$M_{r_1}(\mathbb{C}) \oplus \cdots \oplus M_{r_k}(\mathbb{C})$$

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- Elliott '76: A, B simple AF alg's.

$$A \cong B \iff (K_0(A), K_0(A)_+, [1_A]_0) \cong (K_0(B), K_0(B)_+, [1_B]_0)$$

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Classif. should be in terms of K -theory and related invariants.

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E.g. • $L^\infty(X, \mu)$: amenable, AFD

• $C(X)$: amenable, AFD only if
 $\dim X = 0$.

Generalizations: 90s - 2000s

- Focused ^{in part} on algebras with internal approximations of form $\bigoplus M_{n_i}(C(X_i))$

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- Seems restrictive, but a surprising # of examples are of this form, e.g. A_θ is approximated by $\bigoplus M_{n_i}(C(S^1))$

Successes & challenges

- e.g., able to handle many X_i 's
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- counterexamples to Elliott conjecture (early 00s)

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"regular"
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*: Std. hypothesis, true in all examples, conjecturally automatic.

Strategy, in brief:

(C, Gabe, Schafhauser, Tikunsis, White)

"Lift" classification in vNa setting
and prove existence & uniqueness theorems
for maps in C^* -setting.

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This is just canonical action $\mathbb{Z} \curvearrowright \mathbb{Z}_2 = \varprojlim \mathbb{Z}/2^n \mathbb{Z}$.

Here $K_0 = \mathbb{Z}[\frac{1}{2}]$, $K_1 = \mathbb{Z}$.

Ex. 3 Theorem applies to $C(X) \rtimes \Gamma$ if

- X compact metrizable and $\dim X < \infty$
- $\Gamma \curvearrowright X$ is free
- Γ is an elementary amenable group
(e.g. nilpotent, solvable, linear, ...)

(Kerr - Naryshkin '21)