CLASSIFYING AMENABLE OPERATOR ALGEBRAS

JOINT WORK WITH J. GABE, C. SCHAFHAUSER, A. TIKUISIS, AND S. WHITE

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Introduction

Consider

$$M_2(\mathbb{C}) \subset M_4(\mathbb{C}) \subset M_8(\mathbb{C}) \subset \cdots \subset \bigcup M_{2^n}(\mathbb{C})$$
$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

Think of the elements of $\bigcup M_{2^n}(\mathbb{C})$ as "infinite by infinite matrices" that act on the vector space $\ell^2(\mathbb{N})$.

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These are examples of *operator algebras.* This talk is about classifying them: how to tell them apart.

Two very early examples of classification results:

Murray-von Neumann, 1943

$$\overline{\bigcup M_{2^n}(\mathbb{C})}^{\mathrm{WOT}} \cong \overline{\bigcup M_{3^n}(\mathbb{C})}^{\mathrm{WOT}}$$

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How to distinguish these last two? Associate a group with such algebras that is invariant under isomorphism, called $K_0(-)$. It turns out that

$$K_0\left(\overline{\bigcup M_{p^n}(\mathbb{C})}^{\|\cdot\|}\right) = \left\{\frac{m}{p^n} : m, n \in \mathbb{Z}\right\}.$$

Example: $\mathcal{B}(\mathcal{H})$, bounded operators on a Hilbert space

- algebraic structure: $*$-algebra, $\langle T^* v, w\rangle = \langle v, Tw\rangle$
- analytic structure: $\|T\| = \sup\{\|Tv\| : \|v\| = 1\}$, Banach space.

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C^*-algebras

- $A \subset \mathcal{B}(\mathcal{H})$, closed in $\| \cdot \|$
- A abelian $\rightsquigarrow C(X)$
- "Topological flavor"

von Neumann algebras

- $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, closed in WOT.
- $\mathcal{M}$ abelian $\rightsquigarrow L^\infty(X, \mu)$
- "Measure theoretic flavor"

Example: $C^*(\mathbb{Z})$

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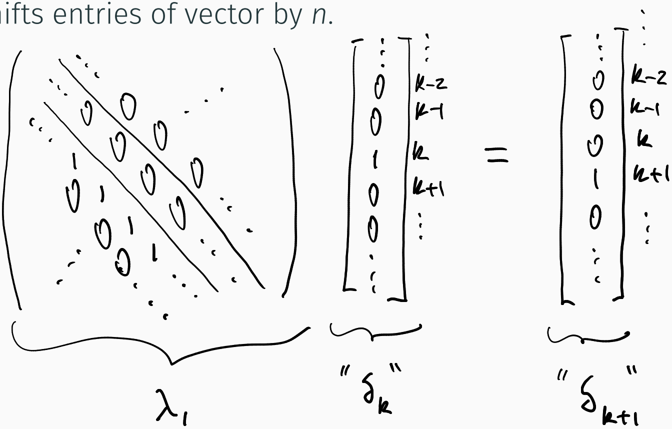

$C^*_\lambda(\mathbb{Z}) :=$ $\|\cdot\|$-closure of $*$-alg. generated by the λ_n's

More general example: group algebras

- Γ: (discrete) group. Get Hilbert space $\ell^2(\Gamma)$ of square summable functions $\Gamma \to \mathbb{C}$ with basis $\{\delta_\gamma\}_{\gamma\in\Gamma}$ (δ_γ: indicator function of $\{\gamma\}$).

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- Fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let $\varphi\colon \mathbb{T} \to \mathbb{T}$ be rotation by $2\pi\theta$.
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- A_θ is built using friendly (even abelian) objects. It's *amenable*.

More general example: group actions/dynamics

- group Γ acts on X (e.g compact metric space) by homeomorphisms: $\Gamma \overset{\alpha}{\curvearrowright} X$.
- Get induced action of Γ on $C(X)$: $\gamma f = f \circ \alpha_\gamma^{-1}$.
- Roughly speaking, can combine $C^*_\lambda(\Gamma)$ and $C(X)$ and form the *crossed product* $C(X) \rtimes \Gamma$.
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FACTORS, FINITE DIMENSIONAL APPROXIMATIONS, AMENABILITY: CLASSIFYING VN ALGEBRAS

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$\Big\}$ $\mathrm{Dim} = [0, 1]$ $\rightsquigarrow$ type II_1

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$\Bigg\}$ projections $\sim [0, \infty]$ $\rightsquigarrow$ type II_∞

Def: Approximately finite dimensional (AFD) vN algebra M

Contains finite dim'l subalgebras $F_1 \subset F_2 \subset \cdots \subset M$ with WOT-dense union.
(Note: finite dim'l $\Leftrightarrow \bigoplus_{k=1}^{N} M_{n(k)}(\mathbb{C})$.)

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One issue: exhibiting internal finite dim'l approximations verifying AFD condition can be difficult.

Would like abstract condition, avoiding concrete internal structural requirements.

Group case

A (discrete) group Γ is *amenable* if it admits a finitely additive left-invariant probability measure on its subsets — a “mean”.

- Includes finite groups, abelian groups
- Closed under direct limits, taking quotients, subgroups, extensions
- Important non-example: free group $\mathbb{F}_n (n \geq 2)$. Related to Banach-Tarski paradox.

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Can define an analog for C^*-algebras and vN algebras. It turns out (with quite some effort) that:

$$\Gamma \text{ amenable} \Leftrightarrow C^*_\lambda(\Gamma) \text{ amenable} \Leftrightarrow \mathrm{vN}(\Gamma) \text{ amenable.}$$

Connes' theorem, 1976

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Theorem (Connes, Haagerup, Murray–von Neumann)

There is a unique amenable factor for each of the types I_n ($n \in \mathbb{N}$), I_∞, II_1, II_∞, III_λ ($0 < \lambda \leq 1$), and the type III_0 factors correspond to certain ergodic flows.

"A triumph of 20th century mathematics" (V.F.R. Jones).

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Led to further breakthroughs in related areas, e.g.:
all free ergodic probability measure preserving actions of countable amenable groups are orbit equivalent (Connes–Feldman–Weiss).

CLASSIFYING C^*-ALGEBRAS

> Since, in principle, a commutative C*-algebra contains all possible information concerning its related compact Hausdorff space, *it ought to be possible to extract topological information ring-theoretically.* Nothing has yet come of this. Possibly the trouble is that the requisite constructions and calculations are beyond the resources of present-day ring theory.
>
> Irving Kaplansky, 1958

Definition

Suppose $1_A \in A$. $K_0(A)$ is the abelian group generated by classes $[p]_0$, where p is any projection in a matrix algebra over A, subject to the relations

- $[p]_0 = [q]_0$ if $p = uv$ and $q = vu$ for some matrices u, v over A
- $[p]_0 + [q]_0 = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right]_0$.

Extension of Atiyah and Hirzebruch's topological K-theory, which concerned itself with the study of vector bundles using algebraic means.

E.g.: When $A = C(X)$, have $K_0(A) \otimes \mathbb{Q} \cong \bigoplus H^{2n}(X; \mathbb{Q})$.

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Theorem (Elliott, 1977)

AF C^*-algebras are classified by their K_0-groups.

The AF condition is much more restrictive on C^*-algebras than on vN algebras. Useful comparison:

- $L^\infty(X, \mu)$: AFD vN algebra
- $C(X)$: AF C^*-algebra $\Rightarrow X$ is zero dimensional

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Nonetheless:

Elliott's classification program (ICM, 1994)

Classify and understand the structure of simple amenable C^*-algebras, in the spirit of Connes, Haagerup.

TOWARDS A CLASSIFICATION (1990S)

- 1990s, 2000s: Progress classifying "higher dimensional" algebras relying on concrete internal structure. Think of internal $\| \cdot \|$-approximations by C^*-algebras of the form $M_n\big(C(X)\big)$.

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- Important early example: every irrational rotation algebra A_θ is proved to be internally approximated by $M_n\big(C(\mathbb{T})\big)$.
- The *purely infinite* case, the analog of type III vN algebras, settled by Kirchberg and Phillips in late 90s.

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- Recall: in the vN algebra setting, amenability is enough for classification. Not so in the C^*-setting. We need to require regularity in addition to amenability to avoid the counterexamples above.

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Theorem

Simple, amenable, and regular C^*-algebras that satisfy the Universal Coefficient Theorem are classified up to isomorphism by their K-theory and traces.

This settles the central classification conjecture in the C^*-setting.

Our approach not only draws inspiration from, but has a direct connection with the classical vN classification techniques.

Irrational rotation algebras

$A_\theta = C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ satisfies the hypotheses. In this case, the K_0 and K_1 groups are both $\mathbb{Z}^2$. The trace portion of the invariant singles out θ, so that $A_\theta \cong A_{\theta'} \Leftrightarrow \theta = \pm\theta' \bmod \mathbb{Z}$.

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Theorem applies to lots of nice actions

Space: $X = \prod_{i=1}^{\infty}\{0,1\}$; action: $+1$ with carry over:

$$(1\,1\,0\,0\ \cdots) \xmapsto{+1} (0\,0\,1\,0\ \cdots) \xmapsto{+1} (1\,0\,1\,0\ \cdots).$$

This is just the canonical action $\mathbb{Z} \curvearrowright \mathbb{Z}_2 = \varprojlim \mathbb{Z}/2^i\mathbb{Z}$.
Here $K_0 = \mathbb{Z}[\frac{1}{2}]$ and $K_1 = \mathbb{Z}$.

More general crossed products (Kerr–Naryshkin)

The classification applies to applies to $C(X) \rtimes \Gamma$ if

- X is a compact metric space of finite covering dimension
- $\Gamma \curvearrowright X$ is free
- Γ is elementary amenable
  (Γ is built up starting with finite or abelian groups; e.g., nilpotent groups, solvable groups, linear groups, . . .)

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Thank you!