TOPOLOGY PRELIMINARY EXAM
AUGUST, 2026

All responses require justification.

(1) Prove or disprove with an example: A connected space is locally connected.

(2) A bijection $f : M \to N$ between two metric spaces (M, ρ) and (N, σ) is called an *isometry* if $\rho(x, y) = \sigma(f(x), f(y))$ for all $x, y \in M$. Show that if f is an isometry then f and f^{-1} are both continuous functions.

(3) Let X and Y be Hausdorff spaces, and let $K \subset X \times Y$ be compact. Prove that there are compact sets $A \subset X$ and $B \subset Y$ such that $K \subset A \times B$.

(4) Let A be a subset of a set X.

   (a) Show that the family of all subsets of X that contain A, together with the empty set, is a topology on X.

   (b) For this topology, if Y is any subset of X, what is its interior?

   (c) For this topology, if Y is any subset of X, what is its closure?

(5) Give an example of two topological spaces with the same fundamental group that are not homotopy equivalent, and explain why.

(6) Consider the 2-dimensional simplex Δ^2. Orient all three of its edges counterclockwise and then identify together all three edges (in other words, form the quotient space obtained by gluing the three edges together). Compute all the homology groups of the resulting space.

(7) Use the Mayer-Vietoris sequence and induction on the dimension n to compute the homology groups of S^n for all $n \geq 0$.

(8) Let $X = S^1 \vee S^2$ be the wedge of the circle and the 2-dimensional sphere. Up to homeomorphism, describe all connected covering spaces of X.