REAL ANALYSIS PRELIMINARY EXAMINATION

AUGUST 2024

Work all 8 problems, which are worth 10 points each. Justification is required for all statements.

(1) Let $\omega = (\tan(z/2) + \sin y)\, dy \wedge dz + (3y - e^{z^2})\, dz \wedge dx + z\, dx \wedge dy$. Compute

$$\int\int_S \omega,$$

where S is the boundary of the solid bounded by the paraboloid $z = 2 - x^2 - y^2$ above and the cone $z = \sqrt{x^2 + y^2}$ below.

(2) Let $(a_1, a_2, \dots)$ be the sequence of real numbers inductively defined by

$$a_1 = 1, \qquad a_n = \frac{1}{1 + a_{n-1}} \quad (n \geq 2).$$

(a) Prove that the subsequence (a_{2k}) is monotonically increasing and the subsequence (a_{2k+1}) is monotonically decreasing .
(b) Prove that the sequence (a_n) converges, and compute its limit.

(3) (a) Suppose that $f : [0, 1] \to \mathbb{R}$ is a Riemann integrable function on $[0, 1]$. Suppose that $g : [0, 1] \to \mathbb{R}$ is another function such that the set $E = \{x \in [0, 1] \mid f(x^2) \neq g(x)\}$ is finite. Prove that g is also a Riemann integrable function on $[a, b]$.
(b) If instead the set E is countably infinite, must g still be Riemann integrable on $[a, b]$?

(4) Let a be any real number which is not an integer.
(a) Find the complex Fourier series for $g(x) = e^{-2\pi i a x}$ on $[0, 1]$.
(b) Prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n + a)^2} = \frac{\pi^2}{\sin^2(\pi a)}.$$

(5) (a) Prove that the infinite series

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{(n+a)^2}$$

defines a continuous function in a over $-1 < a < 1$.

(b) In this part you can assume that the identity in 4b is valid. Prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(6) Assume $f : \mathbb{R} \to \mathbb{R}$ is a continuous non-negative function such that $\int_{-\infty}^{\infty} f(x)\, dx = M < \infty$. Prove that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \left| f\left(x + \frac{1}{n}\right) - f(x) \right| dx = 0.$$

(7) Let f and g be continuous real-valued functions on a compact set K in $\mathbb{R}^n$ with a nonempty interior such that the maximum value of f occurs at a single point p in the interior of K. Prove that for all sufficiently small values $\epsilon > 0$, the function $f + \epsilon g$ has a maximum at an interior point of K.

(8) Consider the set W of all (x, y, u, v) in $\mathbb{R}^4$ satisfying the following system of equations:

$$\begin{cases} u^3 + xv - y = 2, \\ v^3 + yu - x = 0. \end{cases}$$

(a) Is it possible to express the variables (u, v) as uniquely defined functions of the variables (x, y) near the point $(x, y, u, v) = (0, -1, 1, 1)$ in W?

(b) Is it possible to express the variables (x, y) as uniquely defined functions of the variables (u, v) near the point $(x, y, u, v) = (0, -1, 1, 1)$ in W?