Real Analysis Preliminary Exam
January, 2025

Justification is required for all statements.

(1) Suppose that $f : [-\pi, \pi] \to \mathbb{R}$ is a continuously differentiable function such that $f(\pi) = f(-\pi)$ and $f'(\pi) = f'(-\pi)$, and such that
$$\int_{-\pi}^{\pi} f(\theta) \sin(k\theta)\, d\theta = 0$$
for all $k \in \{1, 2, ...\}$. Prove that $f(\theta) = f(-\theta)$ for all $\theta \in [-\pi, \pi]$.

(2) Let M be the upper hemisphere of the sphere of radius R centered at the origin. That is,
$$M = \{(x, y, z) \ : \ z > 0 \text{ and } x^2 + y^2 + z^2 = R^2\}.$$
Let
$$\mathbf{F}(x, y, z) = \left(x^2 e^{y^2 - z^4}\right) \mathbf{i} + \left(e^{x^2+y^2} + yz\right) \mathbf{k}.$$
Find $\int_M \mathbf{F} \cdot \mathbf{n} \, dS$, where $\mathbf{n}$ is the outward pointing unit normal to the surface and dS is the area element.

(3) Let A and B be two compact subsets in $\mathbb{R}^n$. Define
$$A + B = \{a + b \ : \ a \in A, \ b \in B\}.$$
Prove that $A + B$ is a compact subset of $\mathbb{R}^n$.

(4) Using the definition of Riemann integrability, prove that if $F : [0, 1] \to \mathbb{R}$ is continuous, then it is Riemann integrable on $[0, 1]$.

(5) (a) Prove or disprove that the Taylor-Maclaurin series for $\cos(x)$ converges pointwise to $\cos(x)$ on $\mathbb{R}$.
(b) Prove or disprove that the Taylor-Maclaurin series for $\cos(x)$ converges uniformly to $\cos(x)$ on $\mathbb{R}$.
(c) Estimate $\cos(0.1)$ accurate to within 0.0001.

(6) Suppose $g(x) = \sum_{n \geq 1} n e^{-nx}$.
(a) Prove that g is continuous on $(0, \infty)$.
(b) Prove that $\int_1^{\infty} g(x)\, dx$ converges, and evaluate the integral.

(7) Suppose f is a continuous function on $\mathbb{R}$ such that $f(x) \neq x$ for all $x \in \mathbb{R}$.
(a) Prove that either $f(x) > x$ for all $x \in \mathbb{R}$ or $f(x) < x$ for all $x \in \mathbb{R}$.
(b) Let $a_0 > 0$, and define inductively $a_n = f(a_{n-1})$ for all $n \in \mathbb{N}$. Show that the sequence $(a_n)_{n \geq 0}$ is monotone.
(c) Show (a_n) is unbounded.

(8) Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be differentiable. Assume that there is a vector $\mathbf{v} \in \mathbb{R}^n$ and a sequence $\mathbf{0} \neq \mathbf{x_k} \to \mathbf{0}$ such that $\mathbf{F}(\mathbf{x_k}) = \mathbf{v}$ for all k. Prove that $\det \mathbf{F}'(\mathbf{0}) = 0$.