COMPLEX ANALYSIS PRELIMINARY EXAM
JUNE 16, 2025

All responses require justification.

(1) Determine all roots of $z^9 + z^8 + \ldots + z + 1$.

(2) Compute
$$\int_\gamma \frac{z+\pi}{(z-\pi)\,(e^z+1)} dz,$$
if γ is the curve defined by $\gamma(t) = 2\pi e^{it}$ with $0 \le t \le 4\pi$.

(3) Let f be an entire function on the complex plane such that $\mathrm{Re}(f(z)) < 0$ for all $z \in \mathbb{C}$. What functions can f be, and why?

(4) Find the number of solutions to the equation $z^2 e^z = \dfrac{i}{2025}$ inside the open unit disk.

(5) Find all holomorphic functions g defined on $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ such that $|g(z)| \le |z|^{-2}$ and $g(\frac{i}{2}) = 4$.

(6) Let
$$S(z) = \sum_{k=0}^{\infty} a_k z^k$$
for complex z. Denote $\Delta(R) = \{z : |z| < R\}$ and $C(R) = \{z : |z| = R\}$.

(a) Give an example of a set of coefficients a_k such that the series $S(z)$ converges on $\Delta(2)$ and diverges on at least one point of $C(R)$ for $R \ge 2$.

(b) Give an example of a set of coefficients a_k such that the series $S(z)$ converges on $\Delta(2) \cup C(2)$ and diverges on at least one point of $C(R)$ for all $R > 2$.

(7) Let $\phi : D(0,2) \to D(0,5)$ be a holomorphic function, such that $\phi(0) = \phi(1) + 2$. Prove that $\{\mathrm{Re}\,\phi(z) : z \in D(0,2)\}$ is an open interval in the real line.

(8) (a) Let F and G be two entire holomorphic functions, such that $F(w_k)^2 = G(w_k)$ for a bounded sequence $(w_k)_{k\ge1}$ of distinct complex numbers. Prove that $F(z)^2 = G(z)$ for all $z \in \mathbb{C}$.

(b) Prove that the conclusion in (a) is false if the word *bounded* is deleted.