# COMPLEX ANALYSIS PRELIMINARY EXAM
# AUGUST, 2025

**All responses require justification.**

(1) Suppose that $u$ and $v$ are real-valued functions on $\mathbb{R}^2$, and suppose a function $f(x+iy) = u(x,y) + iv(x,y)$ is complex-differentiable at $z_0 = x_0 + iy_0$. Prove that the Cauchy-Riemann equations hold at $z_0$.

(2) Suppose that $f$ is entire and satisfies $|f(z)| < e^{Re(z)}$ for all $z \in \mathbb{C}$. Prove that there exists a constant $c \in \mathbb{C}$ with $|c| < 1$ such that $f(z) = ce^z$.

(3) Let $b > 0$. Find

$$\int_0^\infty \frac{1+\cos(x)}{1+b^2x^2}\,dx\,.$$

(4) Let $\phi(z) = \dfrac{e^{1/z}}{z-1}$. Consider the Laurent series of $\phi$ that is centered at 0 and that converges at $z = -2+4i$. Find the three nonzero terms with the greatest powers of $z$ in the series. Give the open domain of convergence of this series.

(5) Let $C(a,r)$ be the circle of radius $r$, centered at $a$, oriented counterclockwise. Find

$$\int_{C(1,\frac{3}{2})} z\cot(\pi z)\,dz\,.$$

(6) Let

$$F(z) = \int_0^\infty e^{izt+t}\,dt\,.$$

(a) Determine the set of all $z \in \mathbb{C}$ for which the integral above converges.

(b) Compute the integral, and show that $F$ is analytic on the set found in (a). Deduce that $F$ can be analytically continued to a larger set, and find the largest possible domain of its analytic continuation.

(7) If $C(r)$ denotes the positively oriented circle of radius $r$ centered at 0, find

$$\int_{C(59)} \frac{z^{2024}}{z^{2025}+\frac{19}{z}}\,dz\,.$$

(8) Let $c$ be a complex number satisfying $|c| < 1$. Prove $|z+c| \le |1+\bar{c}z|$ if and only if $|z| \le 1$, with equality holding if and only if $|z| = 1$.