COMPLEX ANALYSIS PRELIMINARY EXAM
JANUARY 18, 2012

(1) Suppose that a function $f(z) = u(z) + iv(z)$ with u, v real-valued is analytic in a domain D and $v(z) = [u(z)]^2$ for every $z \in D$. Prove that f is constant on D.

(2) Consider the mapping $f(z) = \sin(z)$. Determine the image of the vertical line $\mathrm{Re}(z) = \frac{\pi}{6}$.

(3) Find the Laurent expansion at zero of the function $f(z) = \dfrac{2}{z^2 - 5z + 6}$ valid for $2 < |z| < 3$.

(4) Use the Cauchy Integral Formula to prove Liouville's Theorem.

(5) Compute

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+2x+2)}\,dx.$$

(6) Find an analytic function $g : A \to B$ that is onto, where $A = \{z \in \mathbb{C} : |z - i| < 1\}$ and $B = \{z \in \mathbb{C} : |z| < 4 \text{ and } \mathrm{Re}(z) > 0\}$.

(7) Suppose that g is a holomorphic function defined on $\{z \in \mathbb{C} : z \neq 0\}$ and that $|g'(z)| \leq \dfrac{1}{|z|^{3/2}}$ for $0 < |z| \leq 1$. Prove that $z = 0$ is a removable singularity.

(8) Suppose that f is a holomorphic function on the open disk of radius 5 centered at 0, and suppose that f maps the closed annulus $\{z : 1 \leq |z| \leq 2\}$ into the open unit disk. Prove that the restriction of f to $D(0,2) = \{z : |z| < 2\}$ has exactly one fixed point.