COMPLEX ANALYSIS EXAM
JUNE 16, 2011

(1) Find the Laurent series at zero of the function $h(z) = \dfrac{1}{z^4 + 8z}$ that converges on the set $\{z : |z| = 3\}$, and find its annulus of convergence.

(2) (a) How many entire functions f satisfy $f(n) = n$ for all $n = 1, 2, 3, \ldots$?

(b) How many entire functions f satisfy $f\left(\frac{1}{2n}\right) = \frac{1}{n^2}$ for all $n = 1, 2, 3, \ldots$?
(Your answer to each part could be 0, a positive integer, or infinitely many.)

(3) Evaluate $\displaystyle\int_0^\infty \frac{\cos(ax)}{x^2 + b^2}\,dx$, where $a, b > 0$.

(4) Prove the (partial) Schwarz Lemma: If $f : D \to D$ is a holomorphic function on the open unit disk D and $f(0) = 0$, then $|f(z)| \le |z|$ for all $z \in D$ and $|f'(0)| \le 1$.

(5) Let F be an entire holomorphic function that has exactly two zeros in the set $\{z \in \mathbb{C} : |z - i - 1| < 4\}$ and no zeros on its boundary. Let α denote the boundary of this set, oriented counterclockwise. Prove that

$$\left|\int_\alpha \frac{zF'(z)}{F(z)}dz\right| < 2\pi\left(\sqrt{8} + 8\right).$$

(6) Find all possible entire functions $a(z)$ such that there exist complex constants c and d such that $|a(z)| \le |cz^2 + d|$ for all z.

(7) Is there an entire function $f(z)$ for which $|z|e^{|z|} \le |f(z)|$ for all z?

(8) Find an angle-preserving map $\phi : A \to B$ from the set $A = \{(x, y) : y > 1\} \subset \mathbb{R}^2$ to the set $B = \{(u, v) : u^2 + v^2 < 4, u < 0\} \subset \mathbb{R}^2$, or prove that no such ϕ exists.