Complex Analysis Preliminary Exam

January 13, 2026

1. Let n be a natural number. Describe all solutions z to the equation $(z+1)^n = (z-1)^n$.

2. Consider the function $u(x,y) = \ln\left(\sqrt{x^2+y^2}\right)$.

   (a) Show that $u(x,y)$ is harmonic on $\mathbb{R}^2 \setminus \{(0,0)\}$.

   (b) Does there exist a holomorphic function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ satisfying $\operatorname{Re}(f(x+iy)) = u(x,y)$? Justify your answer.

3. Find a Laurent series for the function

$$f(z) = \frac{z^2 - 2z + 13}{z-2}$$

   which is convergent on the annulus $\{z : 1 < |z-1|\}$.

4. Let $m, k \in \mathbb{R}$ be constants. Consider the set of points $S = \{a+ib \in \mathbb{C} : b > ma + k\}$. Find a conformal map $f : S \to \{z : |z| < 1, \operatorname{Re}(z) > 0\}$.

5. Let $\alpha \in \mathbb{C}$ satisfy $|\alpha| = 1$. Consider the equation

$$\sin z = \frac{\alpha}{z^2}$$

   for $z \in \mathbb{C} \setminus \{0\}$.

   (a) Prove that for each $k \in \mathbb{Z} - \{0\}$ this equation has exactly one solution inside the vertical strip $|\operatorname{Re}(z) - \pi k| < \frac{\pi}{2}$.

   (b) How many solutions (counted with multiplicities) does this equation have inside the vertical strip $|\operatorname{Re}(z)| < \frac{\pi}{2}$?

6. Let $G = \{z : |\operatorname{Re}(z)| < 1, |\operatorname{Im}(z)| < 1\}$. Suppose that $f(z)$ is holomorphic on G and continuous on $\overline{G}$, and that $f(z) = 0$ whenever $\operatorname{Re}(z) = 1$. Prove that $f(z) = 0$ for all $z \in G$. *Hint: consider* $g(z) = f(z)f(-z)f(iz)f(-iz)$.

7. Show that

$$\int_{-\infty}^{\infty} \frac{x}{(x^2+x+1)^2}\,dx = \frac{-2\pi}{3\sqrt{3}}$$

8. Suppose that $f(z)$ is entire and $e^{f(z)}$ is bounded. Prove that $f(z)$ is constant.