Complex Analysis Exam
January 2025

<u>Notation:</u> $D(a;R) = \{z \in \mathbb{C} : |z-a| < R\}$.

1. Suppose that $f : D(0;1) \to \mathbb{C}$ is holomorphic and that
$$\arg(f(z)) = \pi/4$$
for all $z \in D(0;1)$. Prove that f is constant on $D(0;1)$.

2. Let γ be the counter-clockwise-oriented curve $|z-2| = 3$. Find
$$\int_\gamma \frac{\cos(z) - 3\exp(z)}{\sin(z)} dz.$$

3. Let $R > 0$ and $z_0 \in D(0;R)$. Prove that if $f(z) = \sum_{n\geq 0} a_n(z-z_0)^n$ converges on a region containing $D(0; R + \frac{1}{100})$, then there exists $M > 0$ such that $|a_n| \leq \dfrac{M}{R^n}$ for all $n \geq 0$.

4. Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$.

   (a) Find a holomorphic function $F : \mathbb{H} \to D(0;1)$ which is one-to-one and onto.

   (b) Prove that there is no holomorphic function $G : \mathbb{H} \to \mathbb{C}$ that is one-to-one and onto.

5. Suppose that f is holomorphic and non-zero on the disk $D(0;5)$. Prove that
$$\min\{|f(z)| : z \in \overline{D(0;2)}\}$$
is attained at some point on the boundary of $D(0;2)$.

6. Let $f : D(0;3) \to \mathbb{C}$ be a holomorphic function such that $|f(z)| < 1$ for $|z| = 1$. Use Rouché's Theorem to prove that the equation $f(z) = z$ has a solution in $D(0;1)$.

7. Compute
$$\int_{-\infty}^{\infty} \frac{\cos bx}{x^2 + a^2} dx$$
where $a > 0$ and $b > 0$ are constants.

8. Prove that an entire function f is a polynomial if and only if for every $a \in \mathbb{C}$, the power series representation
$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$
has at least one coefficient equal to zero.