COMPLEX ANALYSIS PRELIMINARY EXAM
JULY 8, 2026

All responses require justification.

(1) Find
$$\int_\gamma \frac{z^5}{\sin(z^2) - z^2}\, dz,$$
where γ is the circle of radius $\frac{1}{4}$ centered at 0, oriented counterclockwise.

(2) Let $f(z) = \dfrac{iz}{z-i}$.

(a) Find the image under f of the circle $|z| = 1$.

(b) Find the image under f of the line $\mathrm{Im} z = -1$.

(3) Suppose D is a domain in $\mathbb{C}$, and suppose that F is a nonzero holomorphic function on D. Prove that $\log|F|$ is harmonic on D.

(4) If $\overline{C}(a, r)$ denotes the circle of radius r centered at a, oriented clockwise, find
$$\int_{\overline{C}(2i,5)} \frac{ze^z + e^z - 1}{ze^z - z}\, dz\,.$$

(5) Let U be a bounded open set in $\mathbb{C}$, and suppose that $h : U \to \mathbb{C}$ is a holomorphic function.

(a) Suppose that $|h(z)| \leq 5$ for all $z \in U$, and suppose $h'(z_0) = 1$ for some $z_0 \in U$. Prove that $|h(z)| < 5$ for all $z \in U$.

(b) Find all holomorphic functions $h : U \to \mathbb{C}$ such that $\mathrm{Re}(e^{h(z)}) = \mathrm{Im}(e^{h(z)})$ for all $z \in U$.

(6) Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\; dx}{x^2 + a^2}$, for $a > 0$.

(7) Let g be an entire function such that $3 \leq |g(z) - z^2| \leq 4$ whenever $|z| = 6$. Find the maximum number of solutions to the equation $g(z) = 0$ on the disk $\Delta(2) = \{z : |z| < 2\}$. Provide an example of a function g such that the maximum number of solutions is achieved.

(8) Let $p(z)$ be a holomorphic function on $D(0,2)^* = \{z \in \mathbb{C} : 0 < |z| < 2\}$. Suppose that $(p(z) - 1)(p(z) - 2) = 0$ has no solutions in $D(0,2)^*$. Prove that there exists $M \in \mathbb{N}$ such that $z^M p(z)$ has a removable singularity at $z = 0$.