# Algebra Preliminary Examination
January 18, 2025

**Justification is required for all statements.**

1. Let $G$ and $H$ be groups, and let $\mathrm{Aut}(H)$ denote the group of isomorphisms of $H$ under composition. Suppose $\theta : G \to \mathrm{Aut}(H)$ is a group homomorphism and define $G_\theta$ to be the set $G \times H$ with the following binary operation:

$$(g,h)(g',h') = (gg', h[\theta(g)(h')])$$

   Prove that $G_\theta$ is a group.

2. (a) Give an example of a nontrivial proper normal subgroup of $S_5$.

   (b) Give an example of a subgroup of $S_5$ that is not normal.

3. Let $G$ be a nonabelian group of order 75. Prove that $G$ has exactly 50 elements of order three.

4. Let $\mathcal{A}$ be the subset of $M_5(\mathbb{C})$ consisting of matrices whose eigenvalues, with multiplicity, are $3, 3, 7, 7, 7$. Define an equivalence relation $\sim$ on $\mathcal{A}$ by decreeing that $A \sim B$ if $A$ and $B$ in $\mathcal{A}$ are similar. How many equivalence classes does this equivalence relation have?

5. Let $S$ be a Hermitian matrix (i.e., a self-adjoint matrix with complex entries) and suppose the trace of $S^2$ is zero.

   (a) Prove that $S$ is the zero matrix.

   (b) Give an example of a nonzero complex matrix $T$ with the property that the trace of $T^2$ is zero.

6. Let $F$ be a field, let $E$ be an extension field of $F$, and suppose $\alpha$ is an element of $E$ whose minimal polynomial has odd order. Prove that $F(\alpha) = F(\alpha^2)$.

7. (a) Determine the splitting field of $x^5 - 2$ over the finite field $\mathbb{F}_3$ with three elements.

   (b) Determine the Galois group of $x^5 - 2$ over $\mathbb{F}_3$, both as a set of automorphisms and as an abstract group.

8. Let $I$ be a nonzero ideal in $\mathbb{Z}[x]$, and let $n$ be the lowest degree of a nonzero element of $I$. Suppose $I$ contains a monic polynomial of degree $n$. Prove that $I$ is a principal ideal.