Algebra Preliminary Exam
Spring 2014

Work all eight problems, justifying your work. Each is worth 10 points. You have four hours.

(1) Let H be a subgroup of a group G. Prove that defining multiplication on cosets by $Hx \cdot Hy = Hxy$ is well-defined if and only if H is normal in G.

(2) Prove that distinct (complex) eigenspaces of an orthogonal matrix are orthogonal.

(3) Prove that every finite integral domain is a field.

(4) For matrices A and B where AB is defined, prove

$$\operatorname{rank} AB \leq \min\{\operatorname{rank} A, \operatorname{rank} B\}.$$

(5) Prove that every finite multiplicative subgroup of a field is cyclic.

(6) Let K be a field and $f_1, f_2 \in K[x]$. Let M be the splitting field of $f_1 f_2$, let $L_1 \subset M$ be the splitting field of f_1, and let $L_2 \subset M$ be the splitting field of f_2. Prove that there exists a monomorphism $\tau : \text{Gal}\,(M, L_1) \to \text{Gal}\,(L_2, K)$.

(7) Let S_n denote the symmetric group and A_n denote the alternating group (the subgroup of even permutations).
(a) Prove that $\{\sigma^2 : \sigma \in S_n\}$ generates A_n.
(b) Prove that $\{\sigma^2 : \sigma \in S_n\} = A_n$ if and only if $n \leq 5$.

(8) Let G be a finite group, let $R^\times$ be the multiplicative group of units in a ring R, and let $\phi : G \to R^\times$ be a nontrivial homomorphism. Prove that $\sum_{g \in G} \phi(g)$ is either 0 or a zero divisor in R.