Algebra Preliminary Examination
January 17, 2026

Justification is required for all statements.

1. Factor (with proof) the polynomial $3x^5 + x^4 + 18x^2 + 15x + 3$ as a product of irreducible polynomials over $\mathbb{Q}$.

2. Let M be an 6×6 matrix with real entries such that $M^3 = 2M^2$.

   (a) Give an example of such a matrix M such that M is not diagonalizable and has at least one nonzero eigenvalue, whose eigenspace has dimension at least three.

   (b) Prove that if M is diagonalizable, then $M^3 = 4M$.

3. Let $A = (a_{ij})$ be an $n \times n$ matrix with real entries, with $n \geq 3$. Suppose that $a_{rs} = 0$ whenever $r \leq s$. Prove using the definition that $A^2 = (b_{ij})$ is a matrix such that $b_{rs} = 0$ whenever $r \leq s + 1$.

4. Let S be the set of matrices of the form $M = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ with $a, b, c \in \mathbb{Z}_3$ and $\det M \neq 0$.

   (a) Prove that S is a group under matrix multiplication, and find a formula for the inverse of $M = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$.

   (b) Prove or disprove that S is abelian.

5. Let $I = (a)$ be an ideal in a principal ideal domain R. Prove that I is maximal if and only if a is irreducible.

6. Suppose that B is a group of order 75 that has a unique subgroup of order 3. Find all possible groups of this type, up to isomorphism.

7. Find the splitting field and Galois group of $x^3 - 2$

   (a) over $\mathbb{Q}$.

   (b) over $\mathbb{R}$.

8. Let F be a field with p elements, with p prime. Prove using basic principles that if $f(x) \in F[x]$ is an irreducible polynomial and α is one root of $f(x)$, then $F(\alpha)$ is the splitting field of $f(x)$ over F.