REAL ANALYSIS PRELIMINARY EXAMINATION
JANUARY 14, 2014

(1) Let f be the function of period 2 that equals $|x|$ on $[-1,1]$.

(a) Find the Fourier series of f.

(b) Use it to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

(2) Suppose that the function $h : [0,1] \to \mathbb{R}$ is continuous and that $h(x)$ is irrational for every $x \in [0,1]$. Prove or disprove that h must be a constant function.

(3) Let C be the oriented curve given by $r(t) = (4 + \cos t, -3 \sin t)$ for $0 \le t \le 2\pi$. Compute the line integral $\int_C (e^x - y^2)dx + (\cos y - 2x)dy$.

(4) Suppose that $\sum_{n=0}^{\infty} a_n$ converges.

(a) Prove or disprove that $\sum_{n=0}^{\infty} (a_n)^2$ must also converge.

(b) Prove or disprove that $\sum_{n=0}^{\infty} (a_n)^3$ must also converge.

(5) Suppose that $f : \mathbb{R} \to \mathbb{R}$ such that $|x - y| < \varepsilon$ implies $|f(x) - f(y)| < \varepsilon^2$. Prove that f is constant.

(6) Consider $f : [a,b] \to \mathbb{R}$.

(a) Prove or disprove: If f is Riemann integrable on $[a,b]$, then so is $|f|$.

(b) Prove or disprove: If $|f|$ is Riemann integrable on $[a,b]$, then so is f.

(7) (a) Prove or disprove that there exists a surjective continuous function $F : B \to H$, where B is an open ball in $\mathbb{R}^2$ and $H = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 + 1 = 0\}$.

(b) Prove or disprove that there exists a surjective continuous function $F : H \to B$, with H and B as in (a).

(8) Let g be a differentiable function on $\mathbb{R}$. Prove that if $g'(x) > g(x) > 0$ for all $x \in \mathbb{R}$, then there is a constant $c > 0$ such that $g(x) \ge cx$ for all $x \in \mathbb{R}$.