COMPLEX ANALYSIS - QUESTIONS

Contents

1. The complex plane

1.1 Write each complex number in trigonometric (polar) form.

(a) $i - \sqrt{3}$

(b) $\frac{i+1}{i-1}$

(c) $-\pi$

(d) $3i + \sqrt{3}$

(e) $-2 + 2i\sqrt{3}$

(f) $\frac{1+i}{\sqrt{3}-i}$

1.2 Find the polar form of $-2\sqrt{3} - 2i$.

1.3 Find all the values of $\sqrt[4]{-\frac{1}{2} - i\frac{\sqrt{3}}{2}}$, and simplify your answers.

1.4 Simplify $(2i+2)^7$.

1.5 Rewrite the number 7^{2-3i} in the form $x+iy$, with $x, y \in \mathbf{R}$.

1.6 Find all possible values of $(2i)^{2+i}$.

1.7 Find all the values of $\sqrt[3]{-\sqrt{2}+i\sqrt{2}}$.

1.8 Express in terms of r, θ, where $z = re^{i\theta}$:

(a) $|z-2+3i|^3 = 27$

(b) $\arg(iz) = \frac{2\pi}{3}$

(c) $|z^2-1| = 2$

1.9 Solve the equation $8z^4 = -iz$, putting the solutions in simplified polar form.

1.10 Solve the equation $z^3 - 8i = 0$, giving the solutions in simplified polar form.

1.11 Solve the equation $z^3 + 4\sqrt{2} + 4i\sqrt{2} = 0$, giving the solutions in simplified polar form.

1.12 Solve the equation $1 - z^2 + z^4 - z^6 + z^8 = 0$.

1.13 Let $n \in \mathbb{N}$. For any w on the unit circle in the complex plane, prove that

$$\mathrm{Re}\,(w^n) = \frac{1+w^{2n}}{2w^n}.$$

1.14 Prove that $Re\,(z\overline{w} + \overline{zw}) \leq 2\,|Re\,(z)\,w|$ for all $z, w \in \mathbf{C}$.

1.15 Prove:

(a) For all $z \in \mathbb{C}$, $|\mathrm{Re} z| + |z| \leq 3\,|z| - |\mathrm{Im}\,(z)|$.

(b) For all $z \in \mathbb{C}$, $|\mathrm{Re} z|^2 + |z|^2 = 2\,|z|^2 - |\mathrm{Im}\,(z)|^2$.

1.16 State and prove the triangle inequality for complex numbers.

1.17 True or False. (Justify)

(a) $\mathrm{Im}\,(z^2) = (\mathrm{Im}\,(z))^2$

(b) $\mathrm{Re}(z - \overline{z}) = 3\mathrm{Im}\,(z + \overline{z})$

(c) $(1-i)^{25} = 4096 - 4096i$

2. Geometry in the complex plane

2.1 Given: If (a, b, c) is a point of the Riemann sphere, and $x + iy$ is the corresponding point on the complex plane through the stereographic projection, the formula

$$(a, b, c) = \frac{1}{x^2 + y^2 + 1}\left(2x, 2y, x^2 + y^2 - 1\right)$$

is satisfied.

(a) Consider the circle that is the intersection of the plane $a + b + c = 1$ with the Riemann sphere. Show that the stereographic projection maps this circle to a line, and find the equation of this line.

(b) Explain geometrically why your answer makes sense.

2.2 Let $F(z) = (2 + i) z^3 + cz - 1$, where c is a fixed complex number.

(a) Is $F : \mathbf{C} \to \mathbf{C}$ a surjective map? (A 1-sentence justification of your response is sufficient.)

(b) Suppose that the set $\{z \in \mathbf{C} : F(z) = F(i)\}$ is the union of two points. Find c.

2.3 Suppose that the plane $x_3 = x_1 - x_2$ is intersected with the Riemann unit sphere. What type of curve is this intersection? Find the image of this curve under the stereographic projection.

2.4 Which part of the complex plane is stretched, and which part of the plane is shrunk under the mapping $g(z) = z(1 - z)$?

3. Topology and analysis in the complex plane

3.1 Determine, with proof, if the sequence $(z_n)_{n\geq 1}$ converges or diverges, when for $n \in \mathbb{N}$,

$$z_n = \frac{(1 - i)^{2n}}{(2 + i)^n}.$$

3.2 In each case, determine if $\lim_{z\to 0} f(z)$ exists.

(a) $f(z) = \frac{z^2}{|z|}$

(b) $f(z) = \frac{\mathrm{Re}(z)^2 + 2|z|^2}{z^2}$

(c) $f(z) = \frac{z}{z\bar{z} + 2}$

3.3 Find the set of all $z \in \mathbb{C}$ where the following functions are continuous.

(a) $\frac{1}{z^4 - 2}$

(b) $\frac{1}{|z|^4 - 2}$

(c) $\frac{1 - z^3}{1 - z^4}$

4. Paths

4.1 Find all possible values of the argument of the complex number $\frac{d}{dt} g(v(t))\big|_{t=0}$, if $g(z) = z^3$ and $v : \mathbf{R} \to \mathbf{C}$ is a curve so that $v'(0) = 2 - i$ and $v(0) = 3 - 2i$. Give your answer in radians (Calculator allowed!).

4.2 Find the image of the curve $\gamma(t) = e^{it} - i$ for $0 \leq t \leq \pi$, and indicate the direction the image is traced.

5. Holomorphic Functions

5.1 Find all points where the complex derivative $\frac{\partial f}{\partial z}$ exists. In each case, also determine if the function is holomorphic. If it is holomorphic, find the domain on which it is holomorphic.

(a) $f(z) = z^2 (1 - \overline{z}^2)$

(b) $f(x + iy) = x(\cos y) e^x - y(\sin y) e^x + i(y(\cos y) e^x + x(\sin y) e^x)$

5.2 Justify:

(a) Explain why a holomorphic function g preserves angles between curves through $z_0 \in \mathbf{C}$, as long as $g'(z_0) \neq 0$.

(b) Give an example that shows that the statement above is false if $g'(z_0) = 0$.

5.3 Find all points where the complex derivative $\frac{\partial f}{\partial z}$ exists. In each case, also determine if the function is holomorphic. If it is holomorphic, find the domain on which it is holomorphic.

(a) $f(z) = 3 + 2iz^2$

(b) $f(z) = |z^2 - 2z + 1|$

(c) $f(z) = \frac{Re(z)}{z^2 + |z|^2}$

(d) $f(x + iy) = x + e^y - ie^{1-y} + iex$

5.4 Prove that $g(z) = \sqrt{|\mathrm{Re} z|\,|\mathrm{Im} z|}$ satisfies the Cauchy-Riemann equations at $z = 0$ and is also complex differentiable there.

5.5 At what points are the functions below holomorphic?

(a) $\frac{1}{(z^3-1)^4}$

(b) $\frac{1}{2+|z|^2}$

5.6 Prove that if $f(z) = u(z) + iv(z)$ is holomorphic with u and v real-valued functions, then

$$f'(z) = v_y + iv_x.$$

6. Complex Series & Power Series

6.1 Determine if each series converges or diverges. Determine the sum, if possible.

(a) $\sum_{n=0}^{\infty} 3(3+4i)^{-n}$.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{((3+4i)n)^2}$.

(c) $\sum_{n=0}^{\infty} \frac{in}{e^{in}}$

6.2 Questions:

(a) Write down an expansion of $k(z) = \frac{1}{z}$ as a power series in $(z-2i)$.

(b) Determine, with justification, the set of all z such that the power series you just found converges to $k(z)$.

6.3 Write an expansion of the form $\sum_{n=0}^{\infty} c_n z^n$ for each of the following, and specify where the expansion is valid.

(a) $\frac{2-3i}{2z+3i}$

(b) $\frac{1}{8+z^3}$

(c) $\frac{1}{(z+2)(z-1)}$

(d) $\frac{1}{1+z+z^2+z^3}$

6.4 Find the radius of convergence of each power series:

(a) $\sum_{m=0}^{\infty} \frac{x^m}{2^m+3^m}$

(b) $\sum_{n=1}^{\infty} \frac{n^{n-1}x^n}{(2n+1)^n}$

(c) $\sum_{n=1}^{\infty} \frac{(n!)^2 x^n}{2^n n^{3+2n}}$

6.5 Determine the values of z for which the following series converge absolutely.

(a) $\sum_{n=1}^{\infty} \frac{3^n}{(z-1)^n}$

(b) $\sum_{n=1}^{\infty} \frac{1-z^n}{z^n}$

7. A cornucopia of holomorphic functions

7.1 Find the multivalued exponent $\left[(1+i)^{-i}\right]$.

7.2 Find all values of $(2i)^{3-i}$, in simplified polar form.

7.3 Find all values of $(i)^{-2i}$, in simplified polar form.

7.4 Find the real and imaginary parts of each function:

(a) e^{e^z}

(b) $\cos(i\overline{z})$

7.5 Prove that $\overline{\cos(z)} = \cos(\overline{z})$.

7.6 Give an example of a nonconstant holomorphic function h such that $\overline{h(z)} = ch(\overline{z})$ for some constant c such that $c \neq 1$, for all z in its domain.

7.7 True or False: If g is a nonconstant entire holomorphic function, then g maps each circle centered at the origin to a line or a circle. (Include justification.)

7.8 Express $\cos(\pi + i) - \sinh(2\pi + i)$ in the form $x + iy$, with $x, y \in \mathbf{R}$.

7.9 Define a function f by

$$f(z) = \left\{ \begin{array}{ll} \frac{1-\cos(z)}{z^2} & \text{if } z \neq 0 \\ \frac{1}{2} & \text{if } z = 0 \end{array} \right\}$$

Prove that f is holomorphic on all of $\mathbb{C}$.

7.10 Let $g(z) = \cos(z) + \frac{1}{4-z^2}$

(a) Find the Taylor series of the form $T(z) = \displaystyle\sum_{m=0}^{\infty} c_m z^m$ for $g(z)$.

(b) Evaluate the 75^{th} derivative of $g(z)$ at $z = 0$.

(c) For which z does the Taylor series converge? [Justify briefly.]

(d) For the values of z found in (b), does $T(z) = g(z)$? [Justify briefly.]

(e) Suppose that $g(z)$ is expanded in a Taylor series of the form $S(z) = \displaystyle\sum_{k=0}^{\infty} b_k (z+2i)^k$. For which values of z is it true that $S(z) = T(z)$?

8. Conformal Mapping

8.1 Give an example of a conformal map from the extended complex plane to itself that is 1-1 and onto and maps 2 to ∞.

8.2 Find a conformal map $\alpha(z)$ from the upper half plane onto the disk of radius 2 centered at the origin such that $\alpha(i) = 0$ and $\arg(\alpha'(i)) = -\pi$.

8.3 Find a conformal map $\alpha(z)$ from the upper half plane onto the disk of radius 3 centered at the origin such that $\alpha(2i) = 0$ and $\arg(\alpha'(2i)) = \pi$.

8.4 (a) Where does the function $h(z) = \frac{z+i}{z-2}$ map the point $z = 1$? (b) What is the magnification of the map h at the point $z = 1$? (c) At what angle does h rotate curves through $z = 1$?

8.5 Find a conformal map from the set $\{(x, y) : x > 0, -x < y < x\}$ to the open unit disk.

8.6 Show that $g(z) = \frac{(1+i)z+(1-i)}{-z-i}$ maps the real axis in $\mathbf{C}$ to a circle centered at the origin. Find the radius of that circle.

8.7 Find and graph the image of the open rectangle $\{(x, y) : 1 < y < 2, 1 < x < 2\}$ under the map $h(z) = e^{i\pi z}$.

8.8 Let $w(z)$ be a linear fractional transformation such that $w(i) = 0$ and such that it maps the lines $y = x$ and $x = 2$ in the complex plane to two other lines.

(a) Is it possible that $w(\infty) = \infty$?

(b) Find an example of such a $w(z)$ so that $w(a) = \infty$ for some $a \in \mathbf{C}$.

(c) For such an example as in (b), is it true that $\{w(z) : z \in \mathbf{C}\} = \mathbf{C}$?

8.9 Let $A = \left\{(x, y) : x > 0 \text{ and } y > \sqrt{3}x\right\} \subset \mathbf{R}^2$, and let D be the open disk of radius 1 in $\mathbf{R}^2$ centered at $(2011, -2011)$. Find an orientation-preserving conformal map from A to D (expressed as a function of $z = x + iy$).

8.10 Let the map $F : \mathbf{C} \to \mathbf{C}$ be defined by $F(z) = 3z^4 - 8iz^3 - 6z^2 - 4i$

(a) Is F an onto map?

(b) Is F a $1 - 1$ map?

(c) Is F an analytic map?

(d) Is F a conformal map?

(e) If α and β are two curves in $\mathbf{C}$ that intersect at an angle $\frac{\pi}{6}$, what are the possible angles that occur where the curves $t \mapsto F(\alpha(t))$ and $t \mapsto F(\beta(t))$ intersect? Give an example for each possibility.

8.11 Find the image of the set $\{(x, y) : 0 < x < 2\}$ under the transformation $G(z) = \frac{2z+1}{z+i}$.

8.12 Find a conformal map from the set $\left\{(x, y) : y > 0, x > 0, y < x\sqrt{3}\right\}$ to the open unit disk.

8.13 Prove or disprove that there is a biholomorphic map $w(z)$ from the closed unit disk to itself such that $w(1) = 1$ and $w(0) = \frac{i}{2}$.

8.14 Find a conformal map $\alpha(z)$ from the upper half plane onto the disk of radius 2 centered at the origin such that $\alpha(i) = 0$ and $\arg(\alpha'(i)) = \pi$.

8.15 Find a 1-1 continuous map from the strip $\{(x, y) : 0 < x \leq 1\}$ onto $\mathbf{C} \setminus \{0\}$ such that its restriction to the interior of the given domain is conformal. Show that the inverse is not continuous on $\mathbf{C} \setminus \{0\}$.

8.16 Suppose A and B are two connected and simply connected open domains in $\mathbf{C}$. Suppose that the origin 0 is not in either domain.

(a) Prove that for arbitrary $z_0 \in A$ and $w_0 \in B$, there exists a holomorphic function $g : A \to B$ such that g is one-to-one and onto, and $g(z_0) = w_0$.

(b) In the previous question, is the function g uniquely determined by the given information?

9. Multifunctions

10. Integration in the complex plane

10.1 Evaluate $\int_L |z|^2 \, dz$ over the directed line segment L connecting the point $2+i$ to $-2+i$.

10.2 Find $\int_C z \cos\left(\frac{\pi z}{2}\right) dz$ over the curve C parametrized by $\gamma(t) = \frac{e^t - t^8 + 1}{e^{t^2}} + i\left(t^7 - t\right)$ for $0 \le t \le 1$.

10.3 Find the following integral two different ways (first by rewriting as a combination of real-valued line integrals, second as a complex integral):
$\int_\alpha \left(3 - z - 2z^2\right) \, dz$, where α is the part of the circle of radius three in the fourth quadrant, oriented clockwise.

10.4 Using the last problem, find $\int_\beta \left(3 - z - 2z^2\right) \, dz$, where $\beta : [0,1] \to \mathbf{C}$ is the curve defined by $\beta(t) = 3\left(2t^3 - 1\right) t^2 - 3i \cos\left(\frac{\pi t}{2}\right)$.

10.5 Find a good upper bound for $F(R) = \left| \int_{C_R} \frac{3z-2}{z^4+1} dz \right|$, where C_R is the circle of radius R, oriented counterclockwise. Use it to show that $\lim\limits_{R\to\infty} F(R) = 0$.

11. Cauchy's Theorem I

11.1 Let $\gamma(w; R)$ denote the circle of radius R centered at $w \in \mathbb{C}$, oriented counterclockwise. Evaluate each of these integrals.

(a) $\int_{\gamma(i;2)} \frac{1}{z+2} dz$

(b) $\int_{\gamma(i;3)} \frac{1}{z+2} dz$

(c) $\int_{\gamma(i;2)} \frac{1}{z^2+2} dz$

(d) $\int_{\gamma(i;3)} \frac{1}{z^2+2} dz$

11.2 Let γ be the directed curve that travels counterclockwise around the boundary of the set $\{z : |z| < 3 \text{ and } \operatorname{Im} z > 0\}$. Using deformation and complex partial fractions, find

$$\int_\gamma \frac{1}{z^2+1} dz.$$

12. Cauchy's Theorem II

12.1 Use the Cauchy Integral Theorem to do this problem.

(a) Prove: If g is an entire holomorphic function and if α and β are two piecewise-smooth curves in $\mathbf{C}$ that start at $0.2-3.4i$ and end at $2.8+7.6i$, then $\int_\alpha g(z)\ dz = \int_\beta g(z)\ dz$.

(b) Prove that the previous statement is false if the word "entire" is removed and if α and β are required to be curves inside the domain of g.

13. Cauchy's Formulae

13.1 Let f be a holomorphic function on all of $\mathbb{C}$. Let h be the function defined by $h(z) = f\left(\frac{1}{z}\right)$.

(a) Prove that h is holomorphic on $\mathbb{C}\setminus\{0\}$.

(b) Prove that if $\lim\limits_{z\to 0} h(z) = 0$, then f and h are constant functions.

13.2 Use the Cauchy Integral Formula to prove Liouville's Theorem.

13.3 Prove the Fundamental Theorem of Algebra.

13.4 True or False: If $\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = f(a)$ for a counterclockwise-oriented circle γ centered at a, then f is holomorphic at a. (Provide justification.)

13.5 Evaluate the following integrals. Let $\gamma(w;R)$ denote the circle of radius R centered at $w \in \mathbb{C}$, oriented counterclockwise.

(a) $\int_{\gamma(0;2)} \frac{\sin(z)}{2z-\pi}dz$

(b) $\int_{\gamma(0;10)} \frac{1}{4z^2+2z+1}dz$

13.6 Suppose that $f(z)$ is entire holomorphic and has the property that $|f(2z)| \leq 2|f(z)|$ for all $z \in \mathbb{C}$. What must be true about f?

13.7 Find the value of $\int_{\gamma(0;1)} \frac{1}{az^2+b}dz$ in terms of the nonzero complex numbers a and b.

14. Power series representation

14.1 Let $h(z) = e^{z^6} - \frac{z^5}{z+2i}$.

(a) Find the Taylor series of the form $T(z) = \sum\limits_{m=0}^{\infty} c_m z^m$ for $h(z)$.

(b) For which z does the Taylor series converge? [Justify briefly.]

(c) For the values of z found in (b), does $T(z) = h(z)$? [Justify briefly.]

(d) Suppose that $h(z)$ is expanded in a Taylor series of the form $S(z) = \sum_{k=0}^{\infty} b_k (z + 2i - 1)^k$. For which values of z is it true that $S(z) = T(z)$?

14.2 Find the radius of convergence of the Taylor series for the real-valued function $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x) = \frac{1}{e^x + 3}$, at the point $x = -1$.

14.3 Find the radius of convergence of the Taylor series of $\frac{z}{16+z^2}$ centered at $z = 0$.

(a) By doing a minimum of calculations.

(b) By computing the series and then finding its radius of convergence from the Cauchy-Hadamard formula.

14.4 Determine if each series converges or diverges. Determine the sum, if possible.

(a) $\sum_{n=0}^{\infty} (2-i)^n (3+i)^{-n}$

(b) $\sum_{n=1}^{\infty} \frac{(i)^n}{((1+i)n)^2}$

(c) $\sum_{n=1}^{\infty} \frac{(i-1)^n}{((3+4i)n)^2}$

(d) $\sum_{n=0}^{\infty} (2-i)^n (1+i)^{-n}$

14.5 Find the radius of convergence of each power series:

(a) $\sum_{m=0}^{\infty} \frac{(x+5)^m}{4+3^m}$

(b) $\sum_{n=0}^{\infty} \frac{n^{n-1}}{(n!)2^n} (x-1)^n$

(c) $\sum_{n=0}^{\infty} \frac{n^{2+n} x^n}{(n!)2^n}$

(d) $\sum_{m=0}^{\infty} \frac{x^m}{4^{-m}+3^m}$

14.6 Find the Taylor series of $\frac{z}{1-2z^2}$ centered at $z = 0$. For which $z \in \mathbf{C}$ does the series converge?

14.7 Find the Taylor series of $\ln(1+z^3)$. For which $z \in \mathbf{C}$ does the series converge?

14.8 Find the Taylor series of $\frac{\ln(1+z)-z}{z}$ centered at $z = 0$. For which $z \in \mathbf{C}$ does the series converge?

14.9 Ponder these questions:

(a) Suppose that $g(x)$ is a smooth, real-valued function with Taylor series (at $x = 0$)

$$\sum_{j=0}^{\infty} \frac{x^j}{j!}.$$

Prove or disprove that $g(x) = e^x$ for every $x \in \mathbf{R}$.

(b) Suppose that $g(z)$ is an analytic function with real-valued Taylor series (at $x = 0$)

$$\sum_{j=0}^{\infty} \frac{x^j}{j!}$$

for $x \in \mathbf{R}$. Prove or disprove that $g(z) = e^z$ for every $z \in \mathbf{C}$.

14.10 True or False (Provide justification.)

(a) If $f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$ is analytic on a region containing $\{z : |z - z_0| \leq R\}$, then there is a positive integer M such that $|a_n| \leq \frac{M}{R^n}$ for all $n \geq 0$.

(b) If $\sum_{n \geq 0} a_n (z - z_0)^n$ converges on $\{z : |z - z_0| < R\}$ and diverges on at least one point of $\{z : |z - z_0| = R\}$, then there is a positive integer M such that $|a_n| \geq \frac{M}{R^n}$ for all $n \geq 0$.

(c) If f is a holomorphic function on an open set U in $\mathbf{C}$, then for every $z_0 \in U$, there is a positive number ρ so that the Taylor series of f centered at z_0 converges uniformly on the set $\{z : |z - z_0| < \rho\}$.

14.11 Find the radius of convergence of the Taylor series for the real-valued function $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x) = \frac{1}{e^x + 2}$, at the point $x = -1$.

15. Zeros of holomorphic functions

15.1 Suppose that g is a holomorphic function on the open unit disk $D(0; 1)$ such that $\mathrm{Re}(g(z)) = \mathrm{Im}(g(z))$ for all $z \in D(0; 1)$. Prove that g is a constant function.

15.2 Find, with proof, the number of zeros z of the polynomial $z^6 + z^2 + 27z + 2$ such that $1 < z\overline{z} < 4$.

15.3 Suppose that f is an analytic function defined on the open unit disk that satisfies $f\left(\frac{1}{n}\right) = \frac{3+2n}{n}$ for all $n \geq 1$. Can you determine $f(i + 1)$ from this information? If so, find $f(i + 1)$; otherwise, explain why it is not possible.

15.4 Suppose that $g : \mathbf{C} \to \mathbf{C}$ is an entire holomorphic function such that $Re(g(z)) = 0$ for all $z \in \mathbf{C}$. Prove that g is a constant function, and find all possible values of this function.

15.5 Find the set of all possible holomorphic functions f on $D(0; 2)$ such that $\left(f\left(\frac{i}{n}\right) - \frac{i}{n}\right)^2 = -\frac{1}{n^2}$. Provide justification that your solution(s) are the only possible solutions.

15.6 Suppose that f is an analytic function defined on the open unit disk that satisfies $f\left(\frac{1}{2n}\right) = \frac{1}{n^2}$ for all $n \geq 1$. Find $f\left(\frac{i+1}{2}\right)$.

15.7 Suppose that $g(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)\exp\left(P_n(z)\right)$ is an entire holomorphic function, where each $P_n(z)$ is a polynomial function of z. Assume the nonzero complex numbers z_j satisfy $z_j \neq z_k$ if $j \neq k$.

(a) Prove that it is possible that g is the zero function.

(b) Prove or disprove from basic principles that it must be true that $\lim_{n\to\infty} |z_n| = \infty$ if g is not the zero function.

(c) Prove or disprove from basic principles that it must be true that $P_n(z)$ is uniquely determined for each n.

15.8 We are given an entire function β such that $|\beta(z)| \leq |z+5|$ for all $z \in \{w \in \mathbf{C} : |w| > 12\}$. Prove that $\beta(z) = C_1 z + C_2$ for every $z \in \mathbf{C}$, for fixed complex numbers C_1 and C_2 with $|C_1| \leq 1$.

15.9 Let $p(z) = z^5 + 5z^3 - 1$. Prove that

(a) p has five simple zeros,

(b) all five zeros of p lie in the disk $\{z : |z| < 3\}$, and

(c) no zeros of p lie in the set $\{z : |z| \leq 2 \text{ and } |Re(z)| > 1\}$.

15.10 Determine the number of solutions to the equation $z^9 = 10z + 5$ in the annulus $1 < |z| < 2$.

16. Holomorphic functions: further theory

16.1 Suppose that h is holomorphic on $\mathbb{C}$ and $\lim_{z\to 0} zh\left(\frac{1}{z}\right)$ exists. What does this imply about h? (Justify.)

16.2 Prove each of the following.

(a) If G is holomorphic on $\mathbb{C}$ and $|G(z) - 3| < 1$ for all z such that $|z| > 2$, then G is a constant funcrtion.

(b) If F is holomorphic on $\mathbb{C}$ and $|F(z) - 3| = 1$ for all $z \in D(0; 2)$, then F is a constant function.

16.3 State the open mapping theorem, and use it to prove the maximum modulus principle.

16.4 Suppose that for z in the circle of radius 4 centered at the origin, the entire holomorphic function g is pure imaginary. Prove that g must be a constant.

16.5 Suppose that the function F is holomorphic on the disk of radius 2 centered at the origin, and F satisfies $|F(z)| \leq |\text{Re}(z+1)|$ for all z such that $1 \leq |z| \leq 2$. What must be true about F? (Justify.)

17. Singularities

17.1 Find the principal part of the Laurent expansion about 0 of each function below.

(a) $\frac{1}{z^2 e^z \cos(z)}$

(b) $\frac{1}{z^3 e^z \cos(z)}$

(c) $\frac{z - z\exp(z)}{1+\exp(z)}$

17.2 Suppose that g is a holomorphic function on $\mathbb{C}$ such that there exists $M > 0$ such that $\left|\frac{z-1}{g(z)}\right| \leq M$ for all $z \in \mathbb{C}$ such that $g(z) \neq 0$.

(a) Prove that if $g(z)$ has a zero, then it is a simple zero at $z = 1$.

(b) Prove that there exists a constant $K \in \mathbb{C}$ such that $g(z) = K(z-1)$.

17.3 Find the Laurent series expansions for the function $g(z) = \frac{1}{(2-z)^2}$ corresponding to all possible annuli of convergence.

17.4 Find the Laurent series of the function $h(z) = \frac{1}{z^3+4z}$ that converges on the set $\{z : |z| = 3\}$.

17.5 Find the annulus of convergence of the Laurent series found in the last problem.

17.6 Locate and classify the singularities in $\mathbb{C}$ of each function below.

(a) $\frac{1}{z(z^2+1)^3}$

(b) $\frac{z-\pi}{\sin z}$

(c) $\frac{1}{z\ e^{1/z}}$

(d) $\left(\frac{1}{z}+4\right)^{-1}\sin\left(\frac{1}{z}\right)$

18. Cauchy's Residue Theorem

18.1 Compute the following (with justification).

(a)
$$\int_{\gamma(0;5)} \frac{1}{e^{2z}\left(z+\log(2)\right)} dz$$

(b)
$$\int_{\gamma(2;3)} \frac{1}{z^4 - 2z^3 + z^2} dz - \int_{\gamma\left(3;\frac{5}{2}\right)} \frac{1}{z^4 - 2z^3 + z^2} dz$$

18.2 Find the following integrals

(a) $\int_\alpha \frac{e^z}{z^3} dz$, where α is the curve defined by $|z+1| = 3$, oriented counterclockwise.

(i) (5 points) Using the Cauchy Integral Formula for derivatives

(ii) (5 points) Using the Residue Theorem

(b) $\int_{\Delta}\left(z^2\sin(1/z)+\frac{e^{z^2}\cos(z)}{z^2}\right)dz$, where Δ is the circle of radius 1 centered at 0, oriented counterclockwise.

18.3 For $r>0$, let $I(r)=\int_{C_r}\left(\frac{2z-3}{(z-i)^2}+\cos(5z)\right)dz$, where C_r is the circle of radius r centered at 0, oriented counterclockwise. Find $I(r)$.

18.4 Evaluate $\int_{|z|=1} z^2\exp\left(\frac{i}{z}\right)\ dz$, where the orientation of the circle is counterclockwise.

18.5 Let β be the curve $\beta(t)=(2\cos(t),-\sin(t))$ for $0\le t\le 2\pi$. Let

$$I=\int_{\beta}\frac{e^{3z}}{(z+1)^2}dz\ .$$

(a) Compute I, using the Cauchy integral formula for derivatives.

(b) Compute I, using the Residue Theorem.

18.6 Find

$$\int_{-\infty}^{\infty}\frac{x^2}{x^4+1}dx.$$

18.7 Find the following integrals

(a) $\int_{\alpha}\frac{e^z}{z^3}dz$, where α is the circle of radius 1, oriented counterclockwise.

(i) Using the Cauchy Integral Formula for derivatives

(ii) Using the Residue Theorem

(b) $\int_{\alpha}\frac{1}{z^2+3z}dz$, where α is the circle of radius 1, oriented counterclockwise.

(c) $\int_{\alpha}\frac{1}{z^3+3z}dz$, where α is the circle of radius 3, oriented clockwise.

(d) $\int_{-\infty}^{\infty}\frac{1}{x^2+1}\ dx$

(i) using $\arctan(x)$

(ii) by using partial fractions

(iii) by using the Residue Theorem

(iv) Show that all answers agree.

(e) $\int_{-\infty}^{\infty}\frac{\cos(x)}{x^2-3x+3}\ dx$

18.8 Using complex analysis methods, compute the following.

(a) Find $\int_0^{\infty}\frac{1}{x^4+2x^2+1}dx$.

(b) Find $\int_0^{\pi}\frac{1}{5+4\cos(\theta)}d\theta$.

(c) Find $\int_0^{\infty}\frac{\cos(x)}{x^2+1}dx$.

(d) Find $\int_0^{\infty}\frac{\ln(x)}{(x^2+1)^2}dx$.

18.9 Suppose that $\int_\alpha \frac{g'(z)}{g(z)}dz = 6\pi i$, for a holomorphic function on a region D containing the simple closed Jordan curve α. Suppose that g has exactly two zeros in the interior of α. Prove or disprove that it is possible that these two zeros are simple.

18.10 Use the residue theorem to solve these questions:

(a) For $p \in \mathbb{R}$, find $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + 2p^2}$.

(b) For $p \in \mathbb{R} \setminus \mathbb{Q}$, find $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{(k + 2p)^2}$.

(c) For $p \in \mathbb{R} \setminus \mathbb{Q}$, find $\displaystyle\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k + 2p)^2}$.

19. Harmonic Functions

19.1 Let u be a harmonic function on a nonempty domain $U \subseteq \mathbb{R}^2$.

(a) Prove that u_y is also a harmonic function on U.

(b) Prove or disprove that if u is bounded on U, then a harmonic conjugate of u is also bounded on U.

(c) Prove that the function f on U (thought of as being a subset of $\mathbb{C}$) defined by $f(x + iy) = u_{xx}(x, y) - i\ u_{xy}(x, y)$ is holomorphic on U.

19.2 A harmonic function $u(z)$ on the unit disk is continuous on the closed unit disk except for a finite number of discontinuities on the boundary. Find such a u that satisfies the given condition.

(a) $u\left(e^{i\theta}\right) = \begin{cases} \pi & \text{if } 0 \le \theta \le \pi \\ 0 & \text{if } -\pi < \theta < 0 \end{cases}$.

(b) $u\left(e^{i\theta}\right) = \begin{cases} \cos(\theta) & \text{if } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$.

19.3 Let S and T be two domains in $\mathbb{C}$, such that there exists a holomorphic function $f : T \to S$. Let u be a harmonic function of $z \in S$. Prove that $u \circ f$ is a harmonic function on T.

20. Infinite Products

20.1 Find the values of z such that the infinite product

$$\prod_{k=0}^{\infty} \left(1 + z^{2k}\right)$$

converges.

20.2 Prove that the infinite product

$$\prod_{k=2}^{\infty}\left(1-\frac{1}{(k+1)(k-1)}\right)$$

converges, and find the limit.

20.3 Prove that the infinite product

$$\prod_{k=0}^{\infty}\left(1+\frac{(-1)^k z^k}{(k^2+1)\,2^k}\right)$$

converges uniformly and absolutely on a closed disk of some radius $R>0$, centered at zero. Is there a largest possible R such that the statement is true?

20.4 Prove that the infinite product

$$\prod_{k=0}^{\infty}\left(1+\frac{(-1)^k z^k}{(k+1)\,2^k}\right)$$

converges uniformly and absolutely on a closed disk of some radius $R>0$, centered at zero. Is there a largest possible R such that the statement is true?

20.5 Prove or disprove that if both $\prod_{k=1}^{\infty}(1+b_k)$ and $\prod_{m=1}^{\infty}(1+c_m)$ converge, then

$$\prod_{k=1}^{\infty}(1+b_k)\prod_{m=1}^{\infty}(1+c_m)=\prod_{k=1}^{\infty}(1+b_k+c_k+b_kc_k)\,,$$

with the right hand side being a convergent product. What happens if the two products converge absolutely?

20.6 Write a complete proof that for all $z\in\mathbb{C}$,

$$\sin(z)=z\prod_{k=1}^{\infty}\left(\frac{\pi^2k^2-z^2}{\pi^2k^2}\right).$$

20.7 Use the above formula to prove that

$$\cos(z)=\prod_{k=1}^{\infty}\left(\frac{\pi^2k^2-4z^2}{\pi^2k^2-z^2}\right).$$

[Hint: need to show that the product converges!]

20.8 Prove or disprove that

$$\prod_{k=1}^{\infty}e^{-z/k^2}$$

converges at each $z\in\mathbb{C}$. Find the largest set on which the product converges uniformly.

20.9 Prove or disprove that

$$\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-z/k}$$

(a) converges absolutely and uniformly on $\mathbb{C}$.

(b) converges absolutely and uniformly on any bounded subset of $\mathbb{C}$.