⑦ $\tan\left(\frac{-4\pi}{3}\right)$

$= -\sqrt{3}$

slope $= \tan = \dfrac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$

⑧ $\arcsin\left(-\frac{\sqrt{3}}{2}\right) =$ angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sin is $-\frac{\sqrt{3}}{2}$ $= \boxed{-\frac{\pi}{3}}$.

⑨ $\frac{d}{dv}\left(v\cos\left(\ln\left(\arctan(v)\right)\right)\right)$

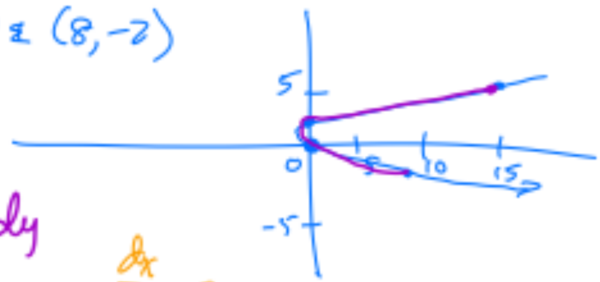
$= \cos(\ln(\arctan(v))) - v\sin\left(\ln(\arctan(v))\right) \cdot \frac{1}{\arctan(v)} \cdot \frac{1}{1+v^2}$

⑩ $\int_0^{\pi}\left(\sin^2(x) + 3^x\right)dx = \int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos(2x) + 3^x\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} + \frac{3^x}{\ln 3}\Big|_0^{\pi}$

$= \frac{\pi}{2} - 0 + \frac{3^{\pi}}{\ln 3} - \left(0 - 0 + \frac{1}{\ln 3}\right)$

$= \boxed{\frac{\pi}{2} + \frac{3^{\pi} - 1}{\ln 3}}$.

⑪ Length of $x = y^2 - 2y$ between $(15, 5)$ & $(8, -2)$

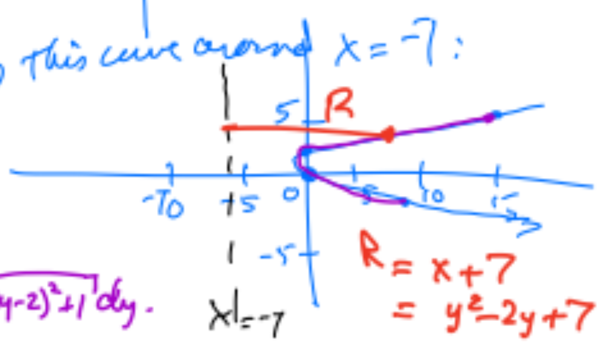


Length $= \int_{y=-2}^{y=5} \sqrt{dx^2 + dy^2} = \int_{-2}^{5} \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy$

$\frac{dx}{dy} = 2y - 2$

$= \boxed{\int_{-2}^{5} \sqrt{(2y-2)^2 + 1}\,dy.}$

⑫ Surface area → rotating this curve around $x = -7$:

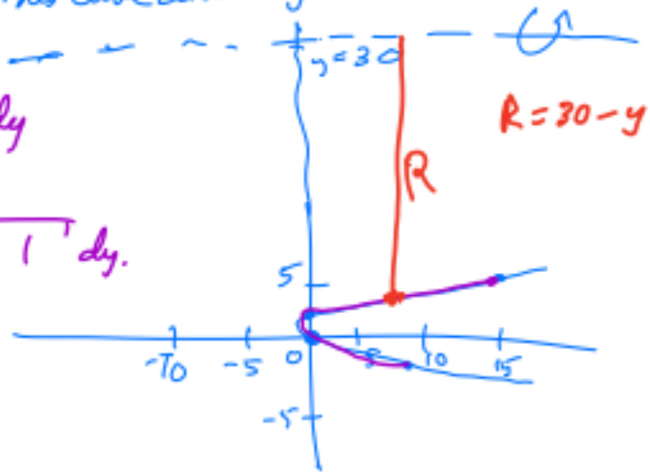


Surface area $= \int_{-2}^{5} 2\pi R\sqrt{(2y-2)^2+1}\,dy$

$= \int_{-2}^{5} 2\pi(y^2 - 2y + 7)\sqrt{(2y-2)^2+1}\,dy.$

$R = x + 7$
$= y^2 - 2y + 7$

(13) Surface area → rotating this curve around $y = 30$

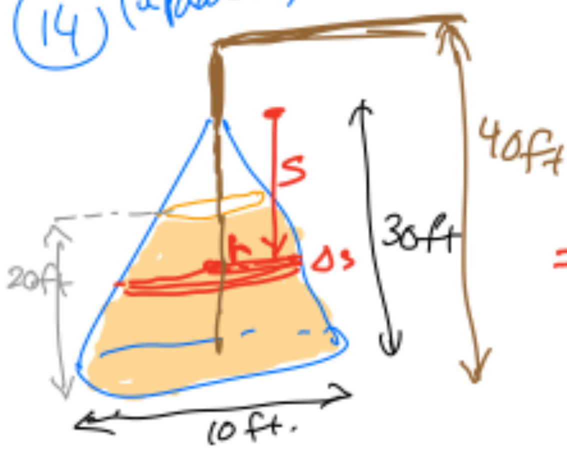


Surface area $= \int_{-2}^{5} 2\pi R\sqrt{(2y-2)^2+1}\,dy$

$$= \int_{-2}^{5} 2\pi(30-y)\sqrt{(2y-2)^2+1}\,dy.$$

$R = 30 - y$

(14) (updated)



Work of Slice

= (Weight of slice) · (Distance of slice)

$$= (\pi r^2)\Delta s \cdot \begin{pmatrix}\text{density}\\ \text{of}\\ \text{water}\end{pmatrix}(s+10)$$

sim triangles

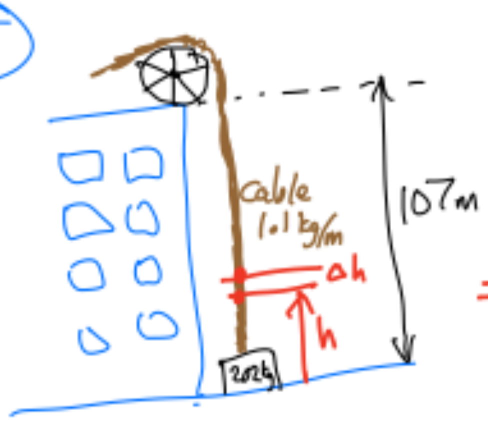
$$\boxed{\frac{r}{s} = \frac{5}{30}} = \frac{1}{6} \Rightarrow r = \frac{s}{6}$$

$\Rightarrow$ Work of slice $= \pi\frac{s^2}{36}(s+10)\,ds$

Total Work $= \displaystyle\int_{s=10}^{s=30} \pi\frac{s^2}{36}(s+10)\,ds\,(62.424)$

in ft. lbs.

(15)



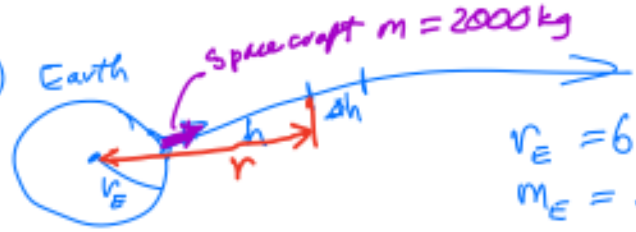
Work req. to go from height h to $h+\Delta h$

= (Weight) · (Distance)

$$= \underbrace{\Big(\underbrace{202\,\text{kg}}_{\text{mass of load}} + \underbrace{1.1\,\frac{\text{kg}}{\text{m}}\cdot(107\text{m}-h)}_{\text{mass of cable}}\Big)\cdot g}_{\text{weight}} \cdot \underbrace{\Delta h}_{\text{Distance}}$$

$$\text{Total Work} = \boxed{\int_{h=0}^{107} \left(202 + 1.1(107-h)\right)\cdot(9.81)\, dh}$$

(16)



$r_E = 6356.8\text{ km} = 6\,356\,800\text{ m} = 6.3568\times10^{6}\text{ m}$

$m_E = 5.9736\times10^{24}\text{ kg}$

Force of gravity

$$F = \frac{G m\, m_E}{r^2}$$

$$G = 6.67428\times10^{-11}\ \frac{Nm^2}{kg^2}$$

Work required to go from h to $h+\Delta h$ = Force · Distance

$$= \frac{G m\, m_E}{(h+r_E)^2}\,\Delta h$$

$$\text{Total Work} = \int_{h=0}^{\infty} \frac{G m\, m_E}{(h+r_E)^2}\, dh$$

$$= \lim_{b\to\infty} \left(G m\, m_E \left(-\frac{1}{(h+r_E)} \right) \right) \Bigg|_0^b$$

$$= \lim_{b\to\infty} G m\, m_E \left[\underbrace{-\frac{1}{(b+r_E)}}_{0} + \frac{1}{r_E} \right] = \frac{G m\, m_E}{r_E}$$

$$= \boxed{\frac{(6.67428\times10^{-11})(2000)(5.9736\times10^{24})}{6.3568\times10^{6}}\ \underbrace{Nm}_{\text{or } J.}}$$

(17)

ⓐ $(2,4,7,11,16,22,\dots) = (a_1, a_2, a_3, \dots)$

Recursive: $a_1 = 2, \quad a_{k+1} = a_k + k + 1$ for $k \geq 1$

Closed form $a_k = \frac{k^2}{2} + \frac{k}{2} + 1$ for $k \geq 1$

a_k	2	4	7	11	16	22
k	1	2	3	4	5	6
k^2	1	4	9	16	25	36
$\frac{k^2}{2}$	.5	2	4.5	8	12.5	
$\frac{k}{2}$	.5	1	1.5	2	2.5	
1	1	1	1	1	1	

ⓑ $\left(\frac{1}{2}, \frac{-3}{4}, \frac{9}{8}, \frac{-27}{16}, \dots\right) = \left(\frac{(1)}{2^1}, \frac{(-1)\cdot 3^1}{2^2}, \frac{(1)\cdot 3^2}{2^3}, \dots\right)$

Recursive $a_1 = \frac{1}{2}, \quad a_k = a_{k-1} \cdot \left(-\frac{3}{2}\right)$

Closed form $a_k = \frac{(-1)^{k+1} 3^{k-1}}{2^k}$ for $k \geq 1$.

(18) Every bounded monotone sequence converges.

(19) See notes from 4/5/23. $\left(3, \frac{3}{2+3}, \frac{3}{2+\frac{3}{2+3}}\right)$ $a_1 = 3/2$ $\quad \frac{3}{2}, \frac{3}{2+\frac{3}{2}}, \frac{3}{2+\frac{3}{2+\frac{3}{2}}}$

(20) ⓐ $\dfrac{3}{2+\boxed{\dfrac{3}{2+\dfrac{3}{2+\dots}}}}$ is the sequence or $a_1 = \frac{3}{2}$

$a_1 = 3, \quad a_k = \frac{3}{2 + a_{k-1}}$ for $k > 1$.

Note that $a_1 = 3 > 0$, $\quad (a_k) = \left(3, \frac{3}{5}, \frac{3}{2+3/5} = \frac{15}{13}, \text{etc}\right)$

$a_k = \frac{3}{2+a_{k-1}} > 0$ if a_{k-1} is, so

by induction $a_k \geq 0$ for all k.

Assuming the limit exists, $\lim a_k = L$ for some $L \in \mathbb{R}$.

$$\lim a_k = \frac{3}{2 + \lim a_{k-1}}$$

$$\Rightarrow L = \frac{3}{2+L}$$ since $\lim a_{k-1} = \lim a_k = L$

$$\Rightarrow L(2+L)=3 \Rightarrow L^2+2L-3=0$$
$$\Rightarrow (L+3)(L-1)=0$$
$$\Rightarrow L=-3 \text{ or } L=1.$$
Since $a_k \geq 0\ \forall k$, $\boxed{L=1.}$

(b) $\sqrt{5+\sqrt{5+\sqrt{5+\cdots}}}$, This is $a_1=\sqrt{5}$, $a_{k+1}=\sqrt{5+a_k}$ for $k\geq 1$.

Note $0<a_1<\sqrt{10}$, and if $0<a_{k-1}<\sqrt{10}$ for some $k\geq 2$,

then $\sqrt{5+0} < \sqrt{5+a_{k-1}} < \sqrt{5+\sqrt{10}}$

$$\Rightarrow 0<\sqrt{5}<a_k<\sqrt{5+5}=\sqrt{10}$$

$\Rightarrow 0<a_k<\sqrt{10}$ also. By induction $0<a_k<\sqrt{10}$ for all k.

Also, (a_k) is increasing.

Proof: $a_2=\sqrt{5+\sqrt{5}} > \sqrt{5+0}=\sqrt{5}=a_1$

Suppose we know $a_k > a_{k-1}$ for some $k\geq 2$.

Then $a_{k+1}^2 = 5+a_k > 5+a_{k-1} = a_k^2$

$\Rightarrow a_{k+1}^2 > a_k^2 \Rightarrow a_{k+1} > a_k$ since both are positive

$\therefore$ By induction, (a_k) is an increasing sequence.

Since (a_k) is increasing and bounded, (a_k) converges to a limit L.

Thus, since $a_k=\sqrt{5+a_{k-1}}$ for $k\geq 2$,
$$\lim a_k = \lim \sqrt{5+a_{k-1}} = \sqrt{5+\lim a_{k-1}}$$
$$L=\lim a_k = \lim a_{k-1} \Rightarrow L=\sqrt{5+L} \Rightarrow L^2=5+L$$
$$\Rightarrow L^2-L-5=0$$
$$\Rightarrow L=\frac{1\pm\sqrt{1+20}}{2}=\frac{1\pm\sqrt{21}}{2}.$$
Since $a_k\geq 0\ \forall k$, $L\geq 0 \Rightarrow \boxed{L=\frac{1+\sqrt{21}}{2}}$

(c) $a_n = \dfrac{2n^2 + \sin(2n)}{2n-1} - n \;, \; n \geq 2.$

Note $a_n = \dfrac{2n^2 + \sin(2n) - n(2n-1)}{2n-1}$

$$= \frac{\cancel{2n^2} - \sin(2n) - \cancel{2n^2} + n}{2n-1}$$

$$= \frac{-\frac{\sin(2n)}{n} + 1}{2 - \frac{1}{n}}$$

Note $\quad -1 \leq -\sin(2n) \leq 1$

$\Rightarrow \quad -\frac{1}{n} \leq \frac{-\sin(2n)}{n} \leq \frac{1}{n}$

$\Rightarrow \quad \frac{-\sin(2n)}{n} \to 0$ as $n \to \infty$

$\therefore \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{-\frac{\sin(2n)}{n} + 1}{2 - \frac{1}{n}} = \boxed{\frac{1}{2}}.$

$\to 0$ $\quad \to 0$

(d) $b_{n+1} = \frac{1}{2} b_n + \dfrac{2}{3 b_{n-1}} + 2^{-5n}$ for $n \geq 2$

$b_1 = 1, \; b_2 = -2.$

$(b_k)_{k\geq 1} = \left(1, -2, -1 + \frac{2}{3} + 2^{-10} \approx -\frac{1}{3}, \approx -\frac{1}{6} - 2 + 2^{-10}, \ldots\right)$

Assuming the limit exists, $\lim\limits_{n\to\infty} b_n = L = \lim\limits_{n\to\infty} b_{n+1} = \lim\limits_{n\to\infty} b_{n-1}$

$\Rightarrow \quad \lim b_{n+1} = \frac{1}{2} \lim b_n + \dfrac{2}{3 \lim b_{n-1}} + \underbrace{\lim 2^{-5n}}_{0}$

$\Rightarrow L = \frac{1}{2} L + \dfrac{2}{3L}$

$\Rightarrow L^2 = \frac{1}{2}L^2 + \frac{2}{3} \Rightarrow \frac{1}{2}L^2 = \frac{2}{3} \Rightarrow L^2 = \frac{4}{3}$

$\Rightarrow L = \pm \frac{2}{\sqrt{3}}$.

Since (b_j) appears to be going negative, we guess that $\boxed{L = -\frac{2}{\sqrt{3}}}$.

21 Find 4th partial sum, determine iff it converges or not, find the sum if possible:

(a) $3 + 4 + \frac{8}{3} + \frac{16}{9} + \frac{32}{27} + \ldots$ || $3+4+\frac{8}{3}+\frac{16}{9} =$ 4th partial sum

$= 3 + \underbrace{4 + 4\cdot\frac{2}{3} + 4\cdot\left(\frac{2}{3}\right)^2 + 4\left(\frac{2}{3}\right)^3 + \cdots}_{\text{geometric series}}$

geometric series
$a =$ first term $= 4$
$r = \frac{2}{3}$

Since $|r|<1$, the geometric series converges, and the sum is $3 + \frac{a}{1-r} = 3 + \frac{4}{1-\frac{2}{3}}$

$= 3 + \frac{4}{1/3} = 3+12 = \boxed{15}$.

(b) $\sum_{k=2}^{\infty} (-1)^k \frac{k}{k-1} =: \sum_{k=2}^{\infty} a_k$

4th partial sum $= \frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4}$

since $\lim a_k \neq 0$ [because $|a_k| = \frac{k}{k-1} \to 1$ as $k \to \infty$],

$\boxed{\text{the series diverges}}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(-1)^{3n}\,5^n} = \frac{4}{-5} + \frac{4^2}{5^2} + \frac{4^3}{-5^3} + \cdots$

4th partial sum $= \frac{-4}{5} + \frac{4^2}{5^2} - \frac{4^3}{5^3} + \frac{4^4}{5^4}$

This is a geometric series with $a = -\frac{4}{5}$, $r = -\frac{4}{5}$.

Since $|r|<1$, the series converges.

the Sum is $\frac{a}{1-r} = \frac{-4/5}{1-(-4/5)} = \frac{-4/5}{9/5} = \boxed{\frac{-4}{9}}$.

(d) $\displaystyle\sum_{m=5}^{\infty} \frac{1}{1000m} = \frac{1}{1000\cdot 5} + \frac{1}{1000\cdot 6} + \frac{1}{1000\cdot 7} + \cdots$

4th partial sum $= \frac{1}{5000} + \frac{1}{6000} + \frac{1}{7000} + \frac{1}{8000}$

$= \frac{1}{1000}\left(\underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots}\right)$

This is the tail of the harmonic series, so it is $= \infty$ (diverges).

$\boxed{\text{Thus, the series diverges (to } \infty \text{ in this case)}}$

(e) $\displaystyle\sum_{p=2}^{\infty} \frac{50p^4\left(3^p + 2^p\right)}{4^p p^4} = \sum_{p=2}^{\infty} \frac{50\cdot(3^p+2^p)}{4^p}$

$\displaystyle = \sum_{p=2}^{\infty} 50\left(\frac{3^p}{4^p} + \frac{2^p}{4^p}\right)$

$\displaystyle = \sum_{p=2}^{\infty} \left(50\cdot\left(\frac{3}{4}\right)^p + 50\left(\frac{1}{2}\right)^p\right)$

4th partial sum

$= \frac{50(3^2+2^2)}{4^2} + \frac{50(3^3+2^3)}{4^3} + \frac{50(3^4+2^4)}{4^4} + \frac{50(3^5+2^5)}{4^5}$

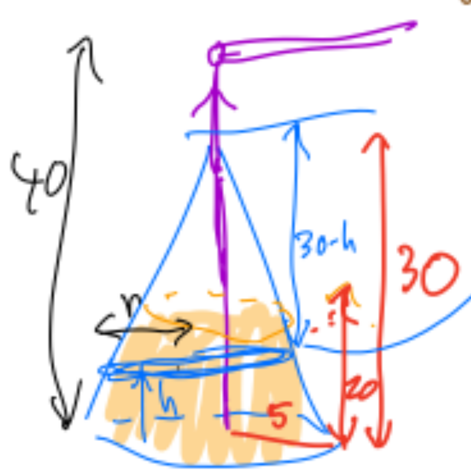
Note $\sum 50\cdot\left(\frac{3}{4}\right)^p$ is a convergent geometric series $\left(r=\frac{3}{4}\right)$, and $\sum 50\cdot\left(\frac{1}{2}\right)^p$ is another convergent geometric series $\left(r=\frac{1}{2}\right)$, so we have

$$\sum_{p=2}^{\infty}\left(50\cdot\left(\frac{3}{4}\right)^p+50\cdot\left(\frac{1}{2}\right)^p\right)=\sum_{p=2}^{\infty}50\left(\frac{3}{4}\right)^p+\sum_{p=2}^{\infty}50\cdot\left(\frac{1}{2}\right)^p$$

$$=\frac{50\left(\frac{3}{4}\right)^2}{1-3/4}+\frac{50\left(\frac{1}{2}\right)^2}{1-1/2}=\frac{50\left(\frac{3}{4}\right)^2}{1/4}+\frac{50\left(\frac{1}{2}\right)^2}{1/2}$$

$$=50\cdot\frac{9}{4}+50\left(\frac{1}{2}\right)=\frac{9\cdot25+50}{2}=\boxed{\frac{275}{2}}.$$

Alternate version of 14



(Work for slice)
$=$ (weight of slice)(distance)
$=$ (Volume of slice $\cdot$ density) $\cdot$ (distance)
$=(\pi r^2\Delta h)(62.424)(40-h)$

Sim Δ's: Small $\frac{r}{30-h}=\frac{5}{30}=\frac{1}{6}$

$\Rightarrow r=\frac{1}{6}(30-h)$.

$$\text{Total Work}=\boxed{\int_{h=0}^{20}\pi\left(\frac{1}{6}(30-h)\right)^2\cdot(62.424)\cdot(40-h)\,dh}$$

④ Definition of Definite Integral.

If f is a piecewise continuous function on $[a,b]$,

then $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x$,

where $\Delta x = \dfrac{b-a}{n}$, $x_k = a + k\,\Delta x$.