# LINEAR ALGEBRA QUESTIONS

## Contents



## 1. Introduction to Matrices

**Problem 1.1.** *Perform each of the following operations, or explain why it is not possible.*

(a) $\begin{pmatrix} 1 & 1 & x \\ 0 & 1 & x \end{pmatrix} + \begin{pmatrix} 2 & 0 & 3 \\ 1 & 5 & 0 \\ 3x & 0 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & x & y \\ 0 & 0 & x \\ 3x+2 & -2 & 3i \end{pmatrix} + \begin{pmatrix} 7 & 4-x & 2 \\ 1 & 1 & 0 \\ 3 & -2i & 3i \end{pmatrix}$

(c) $\begin{pmatrix} 2 & y & x \\ 0 & 0 & x \\ 3x+2 & -2 & 3 \end{pmatrix} - \begin{pmatrix} 7 & 4 & 2 \\ 1 & 1 & 0 \\ 3 & -2 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 7 & 4 \\ 1 & 1 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 2 & x & y \\ 0 & 0 & x \\ 3x+2 & -2 & 3 \end{pmatrix}$

(e) $7\begin{pmatrix} 2 & x & y \\ -1 & x^2 & x \\ 3x+2 & -2 & 3 \end{pmatrix}$

(f) $7y\begin{pmatrix} 2 & x & y \\ -1 & x^2 & x \\ 3x+2 & -2 & 3 \end{pmatrix}$

(g) $(8-3i)\begin{pmatrix} 2 & x \\ 1-i & 0 \\ 3x+2 & -2+i \end{pmatrix} - 2\begin{pmatrix} 7 & 4 \\ 0 & 1 \\ 3 & -2 \end{pmatrix}$

(h) $3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(i) $\begin{pmatrix} 2 & -7 & -2 \end{pmatrix} + \begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix}$

(j) $\begin{pmatrix} 2 & -7 & -2 \end{pmatrix} + \begin{pmatrix} 7 & -1 & 3 \end{pmatrix}$

(k) $\begin{pmatrix} 2 & -7 & -2 \end{pmatrix} \begin{pmatrix} 7 & -1 & 3 \end{pmatrix}$

(l) $\begin{pmatrix} 2 & -7 & -2 \end{pmatrix} \begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix}$

(m) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & -1 & 3 \end{pmatrix}$

(n) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix}$

(o) $\begin{pmatrix} 7 \\ -1 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(p) $\begin{pmatrix} 7 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(q) $\begin{pmatrix} 3 & -1 & 5 \\ x & 1 & 2 \\ -4 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x-1 & 3 \\ 3 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

(r) $\begin{pmatrix} 1 & x-1 & 3 \\ 3 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 3 & -1 & 5 \\ x & 1 & 2 \\ -4 & 1 & 0 \end{pmatrix}$

(s) $\begin{pmatrix} 2 & 3 \\ -1 & 0 \\ 4 & t \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 & 2 \\ -1 & 1 & 1 & 1 \end{pmatrix}$

(t) $\begin{pmatrix} 6 & 2 & 0 & 1 \\ -1 & 1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 3 & 8 \end{pmatrix}$

(u) $\begin{pmatrix} 1 & 1+i & x \\ 0 & 1 & x \end{pmatrix} \begin{pmatrix} 2i & 0 \\ 1+i & -1 \\ 3x & 0 \end{pmatrix}$

**Problem 1.2.** *Solve the following equations for all possible values of the variables in the problem.*

(a) $\begin{pmatrix} x+2 & -2 & 6 \\ 4 & y-1 & 1 \end{pmatrix} = 2 \begin{pmatrix} 3 & -1 & 3 \\ 2 & y & 0.5 \end{pmatrix}$

(b) $\begin{pmatrix} y+2 & -2 & 6 \\ 4 & x-1 & 1 \end{pmatrix} = 2 \begin{pmatrix} 3 & -1 & 3 \\ 2 & x & 1 \end{pmatrix}$

(c) $\begin{pmatrix} y & -2 & 6 \\ 4 & x & 1 \end{pmatrix} + \begin{pmatrix} 2 & y & 3x-y \\ -x & 4 & 8 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 5 \\ -3 & 11 & 9 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & -7 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a \\ 3b \end{pmatrix}$

(e) $\begin{pmatrix} 3 & -7 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -7 \\ 2 \end{pmatrix}$
(f) $\begin{pmatrix} 3 & -7 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -7 \\ -3 \end{pmatrix}$
(g) $\begin{pmatrix} 2 & -4 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$
(h) $\begin{pmatrix} 2 & -4 & 5 \\ 0 & 1 & 3 \\ 2 & -4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$

**Problem 1.3.** *Give a specific example showing that it is possible to find two matrices $A$ and $B$ such that both $AB$ and $BA$ are defined but have different dimensions.*

**Problem 1.4.** *Let $A$ be an $n \times k$ matrix, let $B$ be a $k \times r$ matrix, and let $C$ be an $r \times s$ matrix.*

(a) *Show that $((AB)\,C)_{ij} = \sum_{m=1}^{r} \sum_{p=1}^{k} A_{ip} B_{pm} C_{mj}$ for all possible $i$ and $j$.*
(b) *Use the previous answer to prove the associative property of matrix multiplication: $(AB)\,C = A\,(BC)$.*

**Problem 1.5.** *Find a nonzero matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that*
$M \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

**Problem 1.6.** *Let $A = \begin{pmatrix} -1 & 2 & 7 & 8 & 4 \\ 6 & 2 & 1 & 4 & 5 \\ -2 & 1 & 3 & 0 & 7 \\ -2 & 1 & 1 & 0 & 1 \end{pmatrix}$*

(a) *Find $A_{23}$ .*
(b) *Find $A_{34} - A_{45} + 7A_{32}$ .*
(c) *Find $\sum_{m=2}^{4} A_{m1}$ .*
(d) *Find $\sum_{j=1}^{4} A_{jj}$ .*
(e) *Find $\sum_{j=1}^{4} A_{2j} A_{j3}$*

**Problem 1.7.** *Let $B$ be the $4 \times 4$ matrix defined by the equation $B_{ps} = 2 - p + s$ for all possible $p$ and $s$. Find $B$.*

## 2. Properties of Matrices

**Problem 2.1.** *Prove that if $M$ is any $n \times k$ matrix and $I_k$ is the $k \times k$ identity matrix, then $MI_k = M$.*

**Problem 2.2.** *Let $\heartsuit$ be the special operation on real numbers defined by the equation $a \heartsuit b = a - 2b$.*

(a) *Show that the commutative property does not hold for the operator $\heartsuit$.*
(b) *Show that the associative property does not hold for the operator $\heartsuit$ .*
(c) *Prove that the following distributive property holds for $\heartsuit$ and ordinary multiplication: $c\,(a \heartsuit b) = (ca) \heartsuit (cb)$ for all real numbers $a$, $b$, $c$ .*

**Problem 2.3.** *Simplify the following matrix expressions, if possible, using the properties of matrices. You may assume that the sizes of the matrices are chosen so that all of the* *operations of multiplication, addition, and subtraction can be done. Assume that the capital letters refer to matrices and the lower case letters refer to scalars. Assume that $I$ refers to the appropriate identity matrix.*

(a) $(A+B)-2(B-C)$
(b) $(A+B)C-CB$
(c) $(A+xI)(A-xI)$
(d) $(A+xM)(A-xM)$
(e) $(A+C)A-(A+D)(I+A)$
(f) $(A+B)^3$ *(Note: $M^3$ means $(MM)M$ for matrices $M$.)*

**Problem 2.4.** *Let* $N=\begin{pmatrix}1 & -2 & 5\\ 4 & 1 & 1\\ -7 & -4 & 3\end{pmatrix}$. *Show that it is impossible to find a matrix $A$ so that $AN=I_3$.*

**Problem 2.5.** *Let* $B=\begin{pmatrix}1 & 0 & 6\\ 3 & 2 & 1\end{pmatrix}$. *Show that it is possible to find a matrix $C$ so that $BC=I_2$ , and show that it is not possible to find a matrix $D$ so that $DB=I_3$.*

**Problem 2.6.** *Name the property or definition used in the equation. In this problem, capital letters refer to matrices.*

(a) $AB+AM=A(B+M)$
(b) $A+(-A)=\mathbf{0}$
(c) $(A+B)+C=(B+A)+C$
(d) $A(A+I)+I(A+I)=A(A+I)+(A+I)$
(e) $A(A+I)+I(A+I)=A(A+I)+(IA+I^2)$
(f) $3(M+N)=3M+3N$
(g) $\sum_{j=1}^{k} M_{pj}N_{jp}=(MN)_{pp}$
(h) $(AI)_{rs}=(IA)_{rs}$
(i) $\sum_{a=1}^{b}\left(\sum_{c=1}^{d} M_{ac}\right)N_{ad}=\sum_{a=1}^{b}\sum_{c=1}^{d} M_{ac}N_{ad}$
(j) $\sum_{a=1}^{b}\sum_{c=1}^{d} M_{ac}N_{ca}=\sum_{a=1}^{b}\sum_{t=1}^{d} M_{at}N_{ta}$
(k) $\sum_{a=1}^{b}\sum_{c=1}^{d} M_{ac}N_{ca}=\sum_{a=1}^{b}\sum_{c=1}^{d} N_{ca}M_{ac}$
(l) $\sum_{a=1}^{b}\sum_{c=1}^{d} M_{ac}N_{ca}=\sum_{c=1}^{d}\sum_{a=1}^{b} M_{ac}N_{ca}$
(m) $\sum_{i=1}^{5}\sum_{k=1}^{4}(A_{ik}+B_{ik})C_{k2}=\sum_{i=1}^{5}\sum_{k=1}^{4}(A+B)_{ik}C_{k2}$
(n) $\sum_{i=1}^{5}\sum_{k=1}^{4}(A_{ik}+B_{ik})C_{k2}=\sum_{i=1}^{5}\sum_{k=1}^{4}(B_{ik}+A_{ik})C_{k2}$
(o) $\sum_{k=1}^{p} A_{ak}(5B)_{k4}=\sum_{k=1}^{p} A_{ak}5B_{k4}$
(p) $\sum_{k=1}^{n}\left(\sum_{j=1}^{n} A_{ij}B_{jk}\right)D_{ka}=\sum_{k=1}^{n}\sum_{j=1}^{n} A_{ij}B_{jk}D_{ka}$

**Problem 2.7.** *Prove this version of the second distributive property for matrix multiplication: if $A$ is an $k\times m$ matrix, and if $B$ and $C$ are $m\times p$ matrices, then $A(B+C)=AB+AC$.*

**Problem 2.8.** *Prove that scalar multiplication commutes with matrix multiplication. That is, if $A$ is an $n\times k$ matrix, $B$ is a $k\times s$ matrix, and $x$ is a scalar, then $x(AB)=A(xB)$.*

**Problem 2.9.** *Using the definitions only, prove that for any two $k\times k$ matrices $A$ and $B$,* $\sum_{p=1}^{k}(AB)_{pp}=\sum_{p=1}^{k}(BA)_{pp}$.

**Problem 2.10.** *Simplify the following matrix expression, if possible, using the properties of matrices. You may assume that the sizes of the matrices are chosen so that all of the operations can be done. Assume that the capital letters refer to matrices. Assume that $I$*

*refers to the appropriate identity matrix. You need to show every step, but you do not need to name the property used at each step.*
$(X+Y)(X-Y-I^2)$

**Problem 2.11.** *Name the property or definition used in each equation. Be specific. In this problem, capital letters refer to matrices. Assume that $I$ refers to the appropriate identity matrix.*

(a) $(A+I)A = A^2 + IA$
(b) $(A+(-1)I)A = ((-1)I + A)A$
(c) $\sum_{k=1}^{p} A_{nk}A_{kn} = (A^2)_{nn}$
(d) $2(3B)_{ij} = 2(3B_{ij})$
(e) $\sum_{k=1}^{n}\left(\sum_{j=1}^{n} A_{ij}B_{jk}\right)B_{kn} = \sum_{k=1}^{n}\sum_{j=1}^{n} A_{ij}B_{jk}B_{kn}$

**Problem 2.12.** *Prove the Associate Property of Matrix Addition.*

**Problem 2.13.** *Prove the Inverse Property of Matrix Addition.*

**Problem 2.14.** *Prove that $(x+y)A = xA + yA$ for any $x, y \in \mathbb{C}$ and any $n \times r$ matrix $A$.*

## 3. Gaussian Elimination

**Problem 3.1.** *Determine if each matrix is in reduced row echelon form. If not, explain why not.*

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 0 & 0 & 3 & \frac{1}{3} \\ 0 & 1 & 0 & 2 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$

**Problem 3.2.** *Put each of the following matrices into reduced row echelon form, using elementary row and column operations. Show each step.*

(a) $\begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 3 & -1 & 0 \\ 2 & 0 & 1 & 1 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 10 \\ -5 & 1 \\ 2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -1 & 1 \\ -2 & 2 & 2 \\ 3 & 3 & -3 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -1 & 1 \\ -2 & 2 & 2 \\ 3 & 3 & -1 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 4 & 1 & 50 & 1 \\ 0 & 1 & 3 & -1 & 7 \\ 0 & 0 & 1 & 1 & 8 \end{pmatrix}$

(f) $\begin{pmatrix} 0 & 2 & -3 & 2 & 0 \\ -1 & -2 & 3 & 0 & 0 \end{pmatrix}$

**Problem 3.3.** *Rewrite each system as a matrix equation. Then find the general solution to each system of equations, using Gaussian elimination.*

(a) $$\begin{aligned} 3x - 4y &= 2 \\ x - 2y &= 8 \end{aligned}$$

(b) $$\begin{aligned} 3x - 4y &= 2 \\ x - 2y &= 8 \\ x + y &= -25 \end{aligned}$$

(c) $$\begin{aligned} 3x - 4y &= 2 \\ x - 2y &= 8 \\ x + y &= -24 \end{aligned}$$

(d) $$\begin{aligned} 3x - 4y - z &= 2 \\ x - 2y + z &= 8 \\ 2x - 3y &= 5 \end{aligned}$$

(e) $$\begin{aligned} 3x - 4y - z &= 2 \\ x - 2y + z &= 8 \\ 2x - 3y &= 4 \end{aligned}$$

(f) $$\begin{aligned} a+3d+4e&=6\\ b+2d-e&=3\\ c+d-e&=8 \end{aligned}$$

**Problem 3.4.** *Solve each matrix equation below. (In other words, solve for the unknown matrix.)*

(a) $\begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix} V = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$

(b) $\begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix} V = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}.$

(c) $\begin{pmatrix} 2 & 8 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} M = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$

## 4. Elementary matrices and matrix inverses

**Problem 4.1.** *Find the elementary matrix $E$ that performs the given row operation. Note that you should be able to find this matrix without using reduced row echelon form.*

(a) $E \begin{pmatrix} -1 & 2 & 4 & 1 \\ 2 & 1 & 7 & 0 \\ 0 & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 4 & 1 \\ 2 & 1 & 0 & 14 \\ 0 & 0 & -1 & 2 \end{pmatrix}$

(b) $E \begin{pmatrix} -1 & 2 & 4 & 1 \\ 2 & 1 & 7 & 0 \\ 0 & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 4 & 1 \\ 2 & 1 & 7 & 0 \\ 0 & 0 & -3 & 6 \end{pmatrix}$

(c) $E \begin{pmatrix} -1 & 2 & 4 & 1 \\ 2 & 1 & 7 & 0 \\ 0 & 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 4 & 1 \\ 0 & 0 & -1 & 2 \\ 2 & 1 & 7 & 0 \end{pmatrix}$

(d) $E \begin{pmatrix} 7 & 1 \\ -1 & 2 \\ 0 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -1 & 2 \\ 0 & 1 \\ 7 & 1 \end{pmatrix}$

(e) $E \begin{pmatrix} 7 & 1 \\ -1 & 2 \\ 0 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ -1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

(f) $E \begin{pmatrix} 7 & 1 \\ -1 & 2 \\ 0 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ -1 & 2 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$

**Problem 4.2.** *Find the inverse of each elementary matrix below. Note that you should be able to determine the inverse without any calculations.*

(a) $\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

**Problem 4.3.** *Solve the following matrix equations. (Some of the equations may have no solution.)*

(a) $\begin{pmatrix} 2 & 1 \\ 4 & -1 \\ 2 & 6 \\ 2 & 2 \end{pmatrix} A = \begin{pmatrix} 6 & -\frac{3}{2} \\ 12 & -\frac{9}{2} \\ 6 & 1 \\ 6 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 4 & 1 \\ -2 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3 \end{pmatrix} B = \begin{pmatrix} 7 & 0 \\ -2 & 0 \\ -2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 4 & 1 \\ -2 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3 \end{pmatrix} C = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 6 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 6 & -1 \end{pmatrix} D = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} -3 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(f) $\begin{pmatrix} -3 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} F = \begin{pmatrix} 0 & 4 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & -3 \end{pmatrix}$

**Problem 4.4.** *Find the inverse of each matrix below, or explain why the matrix does not have an inverse.*

(a) $\begin{pmatrix} -3 & 1 & 1 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 2 & 3 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 2 & 4 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} -3 & 7 \\ 1 & -3 \end{pmatrix}$

(f) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(g) $\begin{pmatrix} 2 & -6 & 2 \\ 0 & 1 & 1 \\ 1 & -3 & 2 \end{pmatrix}$

**Problem 4.5.** *Write each matrix below as a product of elementary matrices, or explain why it is not possible.*

(a) $\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix}$

**Problem 4.6.** *If $A$ and $B$ are invertible matrices, prove that $AB$ is invertible and that $(AB)^{-1} = B^{-1}A^{-1}$.*

**Problem 4.7.** *If $A$, $B$, and $C$ are invertible matrices, prove that $ABC$ is invertible and that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.*

**Problem 4.8.** *If $X$ and $Y$ are invertible matrices, prove that $Y^2X$ is invertible and that $(Y^2X)^{-1} = X^{-1}(Y^{-1})^2$.*
*Do not assume knowledge of the formula for $(AB)^{-1}$.*

**Problem 4.9.** *Find all possible solutions to the equation $A^3 = 0$, if $A$ is a $3 \times 3$ matrix.*

**Problem 4.10.** *Find all possible solutions to the equation $M^2 = M$, if $M$ is an invertible $2 \times 2$ matrix.*

**Problem 4.11.** *Find all possible solutions to the equation $M^2 = I$, if $M$ is an invertible $2 \times 2$ matrix.*

**Problem 4.12.** *Assume that all the matrices in the equations below are invertible. In each problem, solve for the matrix $A$. Indicate any assumptions that are needed.*

(a) $AB = BC$
(b) $AB + B^{-1}C = 2A$
(c) $(A - B^{-1})(B + C) = C$
(d) $AMA = A$
(e) $(A^{-1})^2 = BCA^{-1}$
(f) $(A - A^{-1})^2 - A^2 = 5A^{-1}$
(g) $B^2 - I = (B + 2I)(B - 2A)$

**Problem 4.13.** *Solve the systems of equations below, using the inverse of a matrix.*

(a) $\begin{aligned} x + 3y &= 4 \\ 2x + 7y &= 11 \end{aligned}$

(b) $\begin{aligned} x + 3y &= 2 \\ 2x + 7y &= -5 \end{aligned}$

(c) $\begin{aligned} x + y - z &= 8 \\ 2x + y - 3z &= 4 \\ x + z &= 2 \end{aligned}$

(d) $$\begin{aligned} x + y - z &= -8 \\ 2x + y - 3z &= 1 \\ x + z &= 0 \end{aligned}$$

(e) $$\begin{aligned} 2x_2 + x_3 - x_4 &= 2 \\ x_2 - x_3 + 4x_4 &= -3 \\ x_1 - x_3 + 2x_4 &= 1 \end{aligned}$$

## 5. Terminology and additional properties of matrices

**Problem 5.1.** *Find the rank and nullity of each matrix below.*

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 2 \\ -1.5 & -3 \end{pmatrix}$

(d) $\begin{pmatrix} 12 & 9 & 6 \\ -1 & 1 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} 12 & 9 & 6 \\ -1 & 1 & 2 \\ 5 & -1 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 12 & 9 & 6 \\ -1 & 1 & 2 \\ 1 & -1 & -2 \end{pmatrix}$

(g) $\begin{pmatrix} 0 & 2 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -4 & -2 & -2 & 3 \end{pmatrix}$

(h) $\begin{pmatrix} 13 & -4 & 17 & 13 & 14 \\ -2 & 2 & 2 & 8 & 2 \\ 8 & 4 & 10 & 0 & 4 \\ 1 & 6 & 4 & 1 & 0 \end{pmatrix}$

(i) $\begin{pmatrix} 2 & 0 & -1 & 1 \\ -3 & 3 & 0 & 4 \end{pmatrix}$

**Problem 5.2.** *Calculate the transpose of each matrix above, and then calculate the rank and nullity of the transposed matrix.*

**Problem 5.3.** *Suppose that $Q$ is a $3 \times 3$ matrix that is not invertible. What are all the possible values of the rank and nullity of $Q$?*

**Problem 5.4.** *Compute the trace and determinant of each square matrix below.*

(a) $\begin{pmatrix} -1 & 1 \\ 0 & 4 \end{pmatrix}$

(b) $\begin{pmatrix} -4 & 2 \\ 6 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 6 & -1 & 2 \\ 4 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(f) $\begin{pmatrix} 5 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 6 \end{pmatrix}$

(g) $\begin{pmatrix} 5 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & 6 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

(i) $\begin{pmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 8 & 1 \\ 0 & 0 & 6 & 0 \\ -1 & -1 & -14 & -3 \end{pmatrix}$

(j) $\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & -2 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{pmatrix}$

(k) $\begin{pmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & 4 & 1 \\ 0 & 0 & -6 & 0 \\ 0 & 3 & 0 & 1 \end{pmatrix}$

**Problem 5.5.** *Determine which matrices in the previous problem are invertible.*

**Problem 5.6.** *Prove that if $A$ is an $n \times k$ matrix and $B$ is a $k \times r$ matrix, then $(AB)^T = B^T A^T$.*

**Problem 5.7.** *Prove that $(ABCD)^T = D^T C^T B^T A^T$, if the matrix dimensions are such that $ABCD$ is defined.*

**Problem 5.8.** *Prove that if $A$ is an invertible matrix, then $A^T$ is also an invertible matrix, and $\left(A^T\right)^{-1} = (A^{-1})^T$.*

**Problem 5.9.** *Prove that if $A$ and $B$ are $n \times n$ matrices, then $\left((AB)^{-1}\right)^T = (A^{-1})^T (B^{-1})^T$.*

**Problem 5.10.** *Prove that if $A$ and $B$ are square matrices, $\mathrm{tr}\,(A+B) = \mathrm{tr}\,(A) + \mathrm{tr}\,(B)$.*

**Problem 5.11.** *Prove that if $A$ is a square matrix and $c$ is a real number, $\mathrm{tr}\,(cA) = c\mathrm{tr}\,(A)$.*

**Problem 5.12.** *Prove that $\mathrm{tr} A^T = \mathrm{tr} A$ if $A$ is a square matrix.*

**Problem 5.13.** *Prove that if $A$ is an $n \times k$ matrix and $B$ is a $k \times n$ matrix, then $\mathrm{tr}\,(AB) = \mathrm{tr}\,(BA)$.*

**Problem 5.14.** *Prove that it is not always the case that $\det(A+B) = \det A + \det B$ for square matrices $A$ and $B$.*

**Problem 5.15.** *Prove that $\det(AB) = \det(A)\det(B)$ when $A$ and $B$ are $2 \times 2$ matrices, by doing a direct calculation.*

**Problem 5.16.** *Prove that $\det(EA) = \det(E)\det(A)$ when $E$ is an elementary matrix and $A$ is a square matrix.*

**Problem 5.17.** *Suppose that $\det(EA) = \det(A)$ when $E$ is a specific elementary matrix and $A$ is a square matrix. What type(s) of elementary matrix could $E$ be?*

**Problem 5.18.** *Prove that $\det(AB) = \det(A)\det(B)$ when $A$ and $B$ are square matrices.*

**Problem 5.19.** *Prove that $\det\left(A^T\right) = \det(A)$ when $A$ is a $2 \times 2$ matrix, by doing a direct calculation.*

**Problem 5.20.** *Prove that $\det\left(A^T\right) = \det(A)$ when $A$ is a square matrix.*

**Problem 5.21.** *Prove that $\det(A^{-1}) = \frac{1}{\det(A)}$ , if $A$ is an invertible matrix.*

**Problem 5.22.** *Prove that it is not always the case that $\mathrm{tr}\,(A^{-1}) = \frac{1}{\mathrm{tr}(A)}$ if $A$ is an invertible matrix.*

**Problem 5.23.** *Prove that if $c$ is a real number and $A$ is an $n \times n$ matrix, then $\det(cA) = c^n \det(A)$.*

**Problem 5.24.** *Prove that if the determinant of a $2 \times 2$ matrix is zero, then the rank of the same matrix is $1$ or $0$. Give examples of each situation.*

**Problem 5.25.** *Prove that the rank of an $n \times n$ matrix is $n$ if and only if the matrix is invertible.*

**Problem 5.26.** *Prove that the nullity of an $n \times n$ matrix is zero if and only if the matrix is invertible.*

## 6. Properties of Determinants

**Problem 6.1.** *Let* $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ *. Suppose that* $\det(A) = 5$ *. Using this information, find each determinant below.*

(a) $\det \begin{pmatrix} a & b & c \\ d+a & e+b & f+c \\ g & h & i \end{pmatrix}$

(b) $\det \begin{pmatrix} a & b & c \\ g & h & i \\ g & h & i \end{pmatrix}$

(c) $\det \begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix}$

(d) $\det \begin{pmatrix} d & e & f \\ g & h & i \\ a & b & c \end{pmatrix}$

(e) $\det \begin{pmatrix} a & b & b \\ d & e & e \\ g & h & h \end{pmatrix}$

(f) $\det \begin{pmatrix} 2g & 2h & 2i \\ d & e & f \\ a & b & c \end{pmatrix}$

(g) $\det \begin{pmatrix} a & b & c \\ d & e & f \\ 6g & 6h & 6i \end{pmatrix}$

(h) $\det \begin{pmatrix} a & b-2a & c \\ d & e-2d & f \\ g & h-2g & i \end{pmatrix}$

(i) $\det \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$

(j) $\det \begin{pmatrix} a & 0 & 0 \\ d & e & 0 \\ g & h & i \end{pmatrix}$

(k) $\det \begin{pmatrix} a & 0 & c \\ d & 0 & f \\ g & 0 & i \end{pmatrix}$

(l) $\det \begin{pmatrix} a & b & c \\ d & e & f \\ 4d & 4e & 4f \end{pmatrix}$

(m) $\det \begin{pmatrix} a & b & c \\ d & e & f \\ a+2d & b+2e & c+2f \end{pmatrix}$

(n) $\det\begin{pmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{pmatrix}$

**Problem 6.2.** *Using row and column reduction (and* ***not the standard definition****), find the determinants below.*

(a) $\det\begin{pmatrix} 3 & 41 \\ -6 & -85 \end{pmatrix}$

(b) $\begin{vmatrix} -7 & 8 & 4 \\ 0 & 3 & 2 \\ 7 & -9 & 2 \end{vmatrix}$

(c) $\det\begin{pmatrix} 1 & 1 & -1 \\ -4 & 4 & 4 \\ 5 & -5 & 5 \end{pmatrix}$

(d) $\det\begin{pmatrix} 0 & 7 & 2 \\ 14 & 7 & 0 \\ 14 & -7 & 0 \end{pmatrix}$

(e) $\begin{vmatrix} 2 & 1 & 17 & -13 & 2 \\ -4 & 3 & 6 & 71 & 2 \\ 0 & 0 & 11 & 2 & 1 \\ 0 & 0 & 22 & 3 & 8 \\ 0 & 0 & 0 & 0 & 6 \end{vmatrix}$

(f) $\begin{vmatrix} 0 & 0 & 0 & 3 & 902 \\ 0 & 0 & 6 & -85 & 75 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 5 & 23 & 29 & 37 \\ 1 & 67 & 3000 & -35 & -61 \end{vmatrix}$

(g) $\det\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$

(h) $\det\begin{pmatrix} 2 & -1 & 0 & 5 \\ 4 & 1 & 4 & -2 \\ 6 & -3 & 6 & 0 \\ 6 & -3 & 0 & 1 \end{pmatrix}$

**Problem 6.3.** *An* ***upper triangular block matrix*** *is a matrix of the form* $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, *where* $A$ *is an* $r \times r$ *matrix,* $B$ *is an* $r \times s$ *matrix,* $C$ *is an* $s \times s$ *matrix, and* $0$ *stands for the zero matrix of size* $s \times r$*. There is a nice formula for the determinant of such a matrix:* $\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A)\det(C)$*. Prove this formula in the case where* $r = s = 2$*, i.e. where each block is a* $2 \times 2$ *matrix.*

**Problem 6.4.** *Let* $A$*,* $B$*, and* $C$ *be* $n \times n$ *matrices, and suppose that* $\det(A) = 2$, $\det(B) = -3$*, and* $\det(C) = 5$*. Simplify the expression* $\det\left(C\,(AB)^{-1}\,C^{-1}\right)\det\left(A^{-1}\right)$*.*

**Problem 6.5.** *Prove that if a square matrix is not invertible, then its determinant is zero.*

**Problem 6.6.** *Find the determinant of* $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a & b & c & d & e \end{pmatrix}$.

**Problem 6.7.** *We say that $A$ is a **skew-symmetric** matrix if $A^T = -A$ . Prove that every skew-symmetric $7 \times 7$ matrix has zero determinant.*

**Problem 6.8.** *Prove that if $C$ is an invertible $n \times n$ matrix and $A$ is an $n \times n$ matrix, then* $\det(CAC^{-1}) = \det(A)$ .

**Problem 6.9.** *Let $A$, $B$, and $C$ be invertible $n \times n$ matrices. Determine if the statements below are (always) true. If true, justify your answer. If false, give a counterexample.*

(a) $\det(B) = \frac{\det(ABC)}{\det(CA)}$ .
(b) $\det(ABC) = \det(BCA)$ .
(c) $\det(ABC) = \det(BAC)$ .
(d) $\det(ABA^{-1}B^{-1}) = 1$ .
(e) $ABA^{-1}B^{-1} = I$ .
(f) $\det(A - B) = \det(A) - \det(B)$ .
(g) $\det(AB^{-1}) = \frac{\det(A)}{\det(B)}$ .
(h) $\det(A - 3I) = \det(B^{-1}AB - 3I)$ .

**Problem 6.10.** *Let $x_i$, $i = 1, ..., 4$ be positive real numbers with $\sum x_i < 1$. Prove that the following matrix is invertible.*

$$\begin{pmatrix} x_1 - x_1^2 & -x_1x_2 & -x_1x_3 & -x_1x_4 \\ -x_2x_1 & x_2 - x_2^2 & -x_2x_3 & -x_2x_4 \\ -x_3x_1 & -x_3x_2 & x_3 - x_3^2 & -x_3x_4 \\ -x_4x_1 & -x_4x_2 & -x_4x_3 & x_4 - x_4^2 \end{pmatrix}$$

**Problem 6.11.** *Solve the systems of equations below using Cramer's rule, or explain why it is not possible to use it.*

(a) $\begin{cases} x - y = z \\ 2x = z - 4 \\ y + z - x = 2x + 1 \end{cases}$

(b) $\begin{cases} x - y = z \\ 2x = z - 4 \\ 5x - y = 3z - 8 \end{cases}$

(c) $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

**Problem 6.12.** *Find the inverse of each matrix below using Cramer's Rule.*

(a) $\begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

**Problem 6.13.** *Let $A$ be a square matrix of integers. Prove that there exists an inverse $A^{-1}$ with integer entries if and only if* $\det A$ *is* 1 *or* $-1$.

## 7. Linear Algebra meets Geometry: Vectors, Matrices, and linear transformations

**Problem 7.1.** *Consider the quadrilateral with vertices* $(-1,4)$, $(-4,9)$, $(1,6)$, $(-2,11)$

(a) *Show that the quadrilateral is a parallelogram.*
(b) *Find the area of the parallelogram.*

**Problem 7.2.** *Find the area of the triangle with vertices* $(-2,3)$, $(1,7)$, *and* $(-5,2)$.

**Problem 7.3.** *Find the volume of the parallelipiped spanned by the vectors* $(-1,2,4)$, $(-7,1,2)$, $(0,2,1)$.

**Problem 7.4.** *Using determinants, find the area of the quadrilateral with vertices* $(-1,-2)$, $(-2,4)$, $(6,1)$, *and* $(3,-4)$.

**Problem 7.5.** *Find the volume of the tetrahedron with vertices* $(-2,1,4),(3,2,-1),(0,1,4),(3,1,1)$.

**Problem 7.6.** *Find the area of the triangle with vertices* $(0,3)$, $(6,-2)$, $(1,-1)$.

**Problem 7.7.** *Determine if the vectors* $(-1,2,3)$, $(2,4,1)$, *and* $(-6,2,2)$ *are linearly independent. If they are, prove it using the definition. If they are not, find a specific nontrivial linear combination of the vectors that is zero.*

**Problem 7.8.** *Consider the vectors* $(7,0,-1)$, $(8,6,-3)$, *and* $(1,6,-2)$. *Determine if this set of vectors is linearly independent. If it is, prove it using the definition. If it is not, find a specific nontrivial linear combination of the vectors that is zero.*

**Problem 7.9.** *Determine if the vectors* $(-1,2)$, $(2,4)$, *and* $(-6,2)$ *are linearly independent. If they are, prove it using the definition. If they are not, find a specific nontrivial linear combination of the vectors that is zero.*

**Problem 7.10.** *Determine if the vectors* $\begin{pmatrix}-2\\4\\6\end{pmatrix}$, $\begin{pmatrix}1\\1\\1\end{pmatrix}$, *and* $\begin{pmatrix}5\\11\\13\end{pmatrix}$ *are linearly independent. If they are, prove it using the definition. If they are not, find a specific nontrivial linear combination of the vectors that is zero.*

**Problem 7.11.** *Determine if the vectors* $\begin{pmatrix}-2\\4\\6\end{pmatrix}$, $\begin{pmatrix}1\\1\\1\end{pmatrix}$, *and* $\begin{pmatrix}1\\1\\0\end{pmatrix}$ *are linearly independent. If they are, prove it using the definition. If they are not, find a specific nontrivial linear combination of the vectors that is zero.*

**Problem 7.12.** *Determine if the vectors* $(1,2,3,4)$, $(-1,2,3,4)$, *and* $(1,-2,-3,4)$ *are linearly independent. If they are, prove it using the definition. If they are not, find a specific nontrivial linear combination of the vectors that is zero.*

**Problem 7.13.** *Determine if the vectors* $(1,2,3,0)$, $(-1,2,3,0)$, *and* $(1,-2,-3,0)$ *are linearly independent. If they are, prove it using the definition. If they are not, find a specific nontrivial linear combination of the vectors that is zero.*

**Problem 7.14.** *Determine if the vectors* $(1,2,3,0)$, $(-1,-2,3,0)$, *and* $(1,-2,-3,0)$ *are linearly independent. If they are, prove it using the definition. If they are not, find a specific nontrivial linear combination of the vectors that is zero.*

**Problem 7.15.** *Suppose that we have three vectors $v, w, z$ in $\mathbb{R}^3$, and their components satisfy $v_j + 1 = w_j = z_j - 1$ for $j = 1, 2, 3$. Prove that $v, w$, and $z$ are linearly dependent.*

**Problem 7.16.** *Suppose that the vectors $v, w, z$ are graphed in $\mathbb{R}^3$and form a parallelipiped spanned by the vectors. Assume that the parallelipiped is inside the octant $\{(x, y, z) : x > 0, y > 0, z > 0\}$. Find a formula for the diagonal vector inside the parallelipiped that starts from the origin and ends at the opposite corner. Show that it is a linear combination of $v, w, z$.*

**Problem 7.17.** *Consider the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, where $A = \begin{pmatrix} -1 & 4 \\ 1 & 1 \end{pmatrix}$.*

(a) *Find $T_A(e_1)$ and $T_A(e_2)$, where $e_1$ and $e_2$ are the unit vectors in the $x$ and $y$ directions, respectively.*
(b) *Find $T_A(-2e_1 + 3e_2)$, using your last answer.*
(c) *Find a vector $v$ so that $T_A(v) = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$.*

**Problem 7.18.** *Let $M$ be an $n \times n$ matrix. Determine which statements below are equivalent to "$M$ is invertible."*

(a) $\det(M) \neq 0$ .
(b) There exists a matrix $C$ such that $MC = I$ .
(c) The row vectors of $M$ are linearly dependent.
(d) $\text{rref}(M) = I$ .
(e) $\text{rk}(M) = n$ .
(f) $\text{null}(M) = n$ .
(g) $Math = Fun$ .
(h) For every $y \in \mathbb{R}^n$, there is exactly one $x \in \mathbb{R}^n$ that satisfies $Mx = y$ .
(i) $M = E_1 E_2 ... E_k$ for some $n \times n$ elementary matrices $E_1, E_1, E_2, ..., E_k$ .

**Problem 7.19.** *Consider the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$, where $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}$.*

(a) *Find $T_A(e_1)$ and $T_A(e_2)$, where $e_1$ and $e_2$ are the unit vectors in the $x$ and $y$ directions, respectively.*
(b) *Find $T_A(-2e_1 + 3e_2)$, using your last answer.*
(c) *Find a vector $v$ so that $T_A(v) = \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$.*
(d) *Find an example of a vector $w \in \mathbb{R}^3$ so that there does not exist a $v \in \mathbb{R}^2$ such that $T_A(v) = w$.*

**Problem 7.20.** *Give an example of a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.*

**Problem 7.21.** *Consider the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, where $A = \begin{pmatrix} -2 & 4 \\ 3 & -4 \end{pmatrix}$.*

(a) *Find $T_A(e_1)$ and $T_A(e_2)$, where $e_1$ and $e_2$ are the unit vectors in the $x$ and $y$ directions, respectively.*
(b) *Find $T_A(-6e_1 - e_2)$, using your last answer.*

(c) *Find a vector $v$ so that* $T_A(v) = \begin{pmatrix} -9 \\ 10 \end{pmatrix}$, *or prove that no such vector exists. If it exists, is $v$ unique?*

**Problem 7.22.** *Consider the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$, where* $A = \begin{pmatrix} -4 & 1 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}$.

(a) Find $T_A(-3e_1 + 2e_2)$, where $e_1$ and $e_2$ are the unit vectors in the $x$ and $y$ directions, respectively.

(b) Suppose that $w = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ for some $a, b, c \in \mathbb{R}$ and $\det \begin{pmatrix} -4 & 1 & a \\ 2 & 1 & b \\ 0 & 1 & c \end{pmatrix} = 1$. Determine whether or not $w$ is necessarily in the image of $T_A$ , and explain your answer.

**Problem 7.23.** *Consider the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^3$, where* $A = \begin{pmatrix} 1 & -3 \\ 2 & 5 \\ 0 & 1 \end{pmatrix}$.

(a) *Find $T_A(e_1)$ and $T_A(e_2)$, where $e_1$ and $e_2$ are the unit vectors in the $x$ and $y$ directions, respectively.*

(b) *Find $T_A(-2e_1 + 3e_2)$, using your last answer.*

(c) *Find $T_A(-2e_1 + 3e_2)$, using a direct matrix multiplication.*

(d) *Find a vector $v$ so that* $T_A(v) = \begin{pmatrix} 14 \\ 6 \\ -2 \end{pmatrix}$.

(e) *Find an example of a vector $w \in \mathbb{R}^3$ so that there does not exist a $v \in \mathbb{R}^2$ such that $T_A(v) = w$.*

**Problem 7.24.** *Give an example of a nonzero linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that* $f\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

**Problem 7.25.** *Find an example of a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that* $f\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

**Problem 7.26.** *Give an example of a nonzero linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$ such that it is not possible to find $x$ and $y$ so that* $S\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$.

**Problem 7.27.** *Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that there exists a vector $v$ that is not in the image of $T$. Explain why the transformation $T$ must be given by multiplication by a matrix whose determinant is zero.*

**Problem 7.28.** *Consider the vectors $(7, 1, -1)$, $(4, 4, 1)$, and $(1, 0, -1)$.*

(a) *Determine if this set of vectors is linearly independent. If it is, prove it using the definition. If it is not, find a specific nontrivial linear combination of the vectors that is zero.*

(b) *Determine if this set of vectors is linearly independent. Prove your answer using the determinant.*

**Problem 7.29.** *Suppose that $A$, $B$, and $C$ are $r \times r$ matrices, and suppose that the rank of $ABC$ is $r$. Prove that the matrix $B$ is invertible.*

**Problem 7.30.** *Suppose that $a$, $b$, $c$ are real numbers and that* $\det\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \\ -1 & 2 & 1 \end{pmatrix} = 0$. *Let $T_A$ be the linear transformation associated to the matrix* $A = \begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 3 & 1 \end{pmatrix}$.

(a) *Prove or disprove that* $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ *must be in the image of $T_A$.*
(b) *What kind of set is the image of $T_A$ ?*
(c) *Find* $\ker(T_A)$.
(d) *Find* $\operatorname{rk}(A)$ *and* $\operatorname{null}(A)$, *and explain why your answers are consistent with what has been found previously in this problem.*

**Problem 7.31.** *Suppose that $a$, $b$, $c$ are real numbers and that* $\det\begin{pmatrix} 1 & 2 & 3 \\ a & b & c \\ -2 & -4 & -6 \end{pmatrix} = 0$. *Let $T_A$ be the linear transformation associated to the matrix* $A = \begin{pmatrix} 1 & -2 \\ 2 & -4 \\ 3 & -6 \end{pmatrix}$.

(a) *Prove or disprove that* $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ *must be in the image of $T_A$.*
(b) *What kind of set is the image of $T_A$ ?*
(c) *Find* $\ker(T_A)$.
(d) *Find* $\operatorname{rk}(A)$ *and* $\operatorname{null}(A)$, *and explain why your answers are consistent with what has been found previously in this problem.*

**Problem 7.32.** *True or False (Justification not necessary)*

(a) If $C$ and $D$ are $n \times n$ matrices and if the product $DC$ is invertible, then also $C$ is invertible.
(b) If $QR$ is an $m \times m$ matrix that is invertible, then $(QR)^{-1} = Q^{-1}R^{-1}$ if $Q$ and $R$ are invertible.

(c) If $V$ is a 3-dimensional vector space, then every set of 3 vectors in $V$ spans $V$.
(d) If $\det(M) = 0$, then the column vectors of the matrix $M$ are linearly independent.
(e) The number of zero rows in the reduced row echelon form of a matrix is never larger than the number of columns in the matrix.
(f) If $G$ and $H$ are $r \times r$ matrices, then $\det(G + H) = \det(G) + \det(H)$.

## 8. Vector Spaces

**Problem 8.1.** *Let $(V, \oplus, \odot)$ be the set $\mathbb{R}^2$ with the operations of addition and scalar multiplication given by the formulas*
$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + 2y_1, x_2 - 2y_2)$ *and for $c \in \mathbb{R}$,* $c \odot (x_1, x_2) = (cx_1, cx_1)$.

(a) *Prove that $(V, \oplus, \odot)$ satisfies closure for scalar multiplication.*
(b) *Prove or disprove that $(V, \oplus, \odot)$ satisfies the associative property of addition.*

(c) *Prove or disprove that $(V, \oplus, \odot)$ satisfies the associate property of scalar multiplication.*

**Problem 8.2.** *Let $V$ be the set $(-\infty, 0]$ of nonpositive real numbers under the standard operations of addition and scalar multiplication.*

(a) *Prove that $V$ satisfies the commutative property of addition.*
(b) *Find two of the vector space properties that are not satisfied for $V$, and prove that they are not satisfied.*

**Problem 8.3.** *Prove that the set $V = \{g : \mathbb{R}^2 \to \mathbb{R} : g(x_1, x_2) = g(x_1, 0)\}$ with the operations of function addition and scalar multiplication of functions satisfies the following properties of vector spaces:*

(a) *Closure for addition*
(b) *Inverse property of addition*
(c) *Commutative property of addition*

**Problem 8.4.** *Prove or disprove that $W = \{(x, y, z) : 2(x+1) - 3(y-1) = 5(z+1)\}$ is a subspace of $\mathbb{R}^3$.*

**Problem 8.5.** *Prove or disprove that $W = \{(x, y, z) : 2(x-1) - 3(y+1) = 5(z+1)\}$ is a subspace of $\mathbb{R}^3$.*

**Problem 8.6.** *Prove that the zero vector (additive identity) in a vector space is unique.*

**Problem 8.7.** *Prove that in a vector space $V$, for any $v \in V$, $0v$ is the zero vector (additive identity).*

**Problem 8.8.** *Let $V$ be a vector space, and let the three vectors $v, w, z \in V$ and a scalar $c \in \mathbb{R}$ be given. Prove that*

$$c(v + w + z) = cv + cw + cz.$$

**Problem 8.9.** *Given a vector space $V$ and a nonzero vector $v \in V$, suppose that $c_1 v = c_2 v$ for some $c_1, c_2 \in \mathbb{R}$. Prove that $c_1 = c_2$.*

**Problem 8.10.** *Let $w$ be a vector in $\mathbb{R}^3$, and let $w \cdot v$ denote the ordinary dot product of vectors $v, w \in \mathbb{R}^3$.*

(a) *Prove or disprove that the set $S_w = \{v \in \mathbb{R}^3 : v \cdot w = 1\}$ is a subspace of $\mathbb{R}^3$.*
(b) *Prove or disprove that the set $S_w = \{v \in \mathbb{R}^3 : v \cdot w = 0\}$ is a subspace of $\mathbb{R}^3$.*

**Problem 8.11.** *Let $(C, \oplus, \odot)$ be the set $C = \{(x, y) : x^2 + y^2 = 1\}$, and let the operations $\oplus, \odot$ be defined by*

$$\begin{aligned} (x, y) \oplus (u, v) &= (xu - yv, xv + yu) \\ c \odot (x, y) &= (\cos(c\theta), \sin(c\theta)), \end{aligned}$$

*where $\theta$ is the angle in $(-\pi, \pi]$ in radians such that $(x, y) = (\cos(\theta), \sin(\theta))$.*

(a) *Prove that $(C, \oplus, \odot)$ satisfies the identity property of addition. What is the additive identity?*
(b) *Prove that $(C, \oplus, \odot)$ satisfies the commutative property of addition.*
(c) *Prove that $(C, \oplus, \odot)$ is not a vector space.*

**Problem 8.12.** *Let $V = \{f : \mathbb{R} \to \mathbb{R}\}$ be the vector space of all real-valued functions of one variable, using the ordinary definitions of function addition and scalar multiplication of functions. Let $W = \{$differentiable functions $h : \mathbb{R} \to \mathbb{R}$ such that $h'(x) = x^2 h(x) + 5h(x)\}$. Prove that $W$ is a subspace of $V$.*

**Problem 8.13.** *Let $Q$ be the set of polynomials with real number coefficients of degree three or less, meaning that*

$$Q = \left\{p(x) = ax^3 + bx^2 + cx + d : a, b, c, d \in \mathbb{R}\right\}.$$

*Prove that $Q$ is a subspace of the vector space of all polynomials.*

**Problem 8.14.** *Show that $\{(1,2,3), (1,2,4)\}$ is a basis for the subspace $S = \{(x,y,z) : y = 2x\} \subset \mathbb{R}^3$.*

**Problem 8.15.** *Prove that $\{(1,0,3,1), (2,0,5,2)\}$ is not a basis for the subspace $P = \{(x_1,x_2,x_3,x_4) : x_2 = 0\}$.*

**Problem 8.16.** *Prove that $\{(1,2), (3,8)\}$ is a basis for $\mathbb{R}^2$.*

**Problem 8.17.** *Prove that $\{(1,1,-1,1), (-2,-2,2,1)\}$ is a basis of the vector space $V = \{(x_1,x_2,x_3,x_4) : x_1 = x_2$ and $x_3 = -x_2\}$.*

**Problem 8.18.** *Show that $\{(1,2,3), (1,3,4), (1,5,6)\}$ does not span $\mathbb{R}^3$.*

**Problem 8.19.** *Show that $\{(1,2,3), (1,3,4), (1,5,6), (0,1,2)\}$ spans $\mathbb{R}^3$.*

**Problem 8.20.** *Suppose that $\{v_1, v_2, v_3, v_4, v_5\}$ is a set of five vectors in $\mathbb{R}^3$. Prove that this set spans $\mathbb{R}^3$ if and only if the matrix $M$ whose rows are the vectors $v_1, v_2, v_3, v_4, v_5$ satisfies*

$$\operatorname{rref}(M) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

**Problem 8.21.** *Find a basis for the space $\{(a,b,c,d) : a + b = c + d\} \subset \mathbb{R}^4$.*

**Problem 8.22.** *Let $\mathcal{D}$ be the vector space of twice differentiable functions $g : \mathbb{R} \to \mathbb{R}$, and let $\mathcal{F}$ be the vector space of all functions $h : \mathbb{R} \to \mathbb{R}$. Define the function $T : \mathcal{D} \to \mathcal{F}$ by $T(g) = \frac{d^2 g}{dx^2}$ (ie the second derivative).*

(a) *Show that $T$ is a linear transformation.*
(b) *Find* $\ker(T)$.

**Problem 8.23.** *Find a basis for the subspace $\{(x,y) : x = 5y\} \in \mathbb{R}^2$.*

**Problem 8.24.** *Find a basis for* $\ker(T_A)$*, where $A$ is the matrix $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 7 & 1 \end{pmatrix}$.*

**Problem 8.25.** *Consider the matrix .*

(a) $C = \begin{pmatrix} -1 & 0 & 4x \\ 2 & x & 1 \\ 3 & 0 & 1 \end{pmatrix}.$
(b) Find $\det(C)$ by expanding along the last row.
(c) Find all possible values of $x$ so that the nullity of $C$ is one.

**Problem 8.26.** *Let $P$ denote the set of positive real numbers, and let the operations $\oplus$ and $\odot$ be defined for all $x, y \in P$ and $c \in \mathbb{R}$ by*

$$x \oplus y = xy, \ c \odot x = x^c.$$

*It turns out that $P$ is a vector space with these operations ($\oplus$ for addition and $\odot$ for scalar multiplication. A good exercise would be to prove that all ten properties hold. In this problem, specifically show these properties:*

(a) *Closure for addition*
(b) *Existence of additive identity (What is the "zero vector"?)*
(c) *Closure for scalar multiplication*
(d) *Associative property of scalar multiplication*
(e) *Both distributive properties*

**Problem 8.27.** *Suppose that in the last problem, $P$ included the number 0. Is $P$ still a vector space, or does something go wrong?*

**Problem 8.28.** *Let $T : V \to W$ be a linear transformation. Prove that $\mathrm{Im}\,(T)$ is a vector space.*

**Problem 8.29.** *Let $A$ and $B$ both be subspaces of a vector space $W$.*

(a) *Prove or disprove that $A \cap B$ is always a subspace of $W$. (Recall that the intersection $A \cap B$ is defined to be $A \cap B = \{y : y \in A \text{ and } y \in B\}$ .)*
(b) *Prove or disprove that $A \cup B$ is always a subspace of $W$. (Recall that the union $A \cup B$ is defined to be $A \cup B = \{y : y \in A \text{ or } y \in B\}$ .)*

**Problem 8.30.** *Let $\mathcal{F}$ be the vector space of real-valued functions on $\mathbb{R}$. Prove that the function $T : \mathcal{F} \to \mathbb{R}$ defined by $T(g) = g(4)$ is a linear transformation.*

**Problem 8.31.** *Let $K$ be the set of $3 \times 3$ real matrices $M$ such that $M^T = -M$. Prove that $K$ is a subspace of the vector space of all $3 \times 3$ matrices.*

**Problem 8.32.** *Let $W_1$ and $W_2$ be finite-dimensional subspaces of a vector space $V$. Prove that*

$$\dim (W_1 + W_2) = \dim (W_1) + \dim (W_2) - \dim (W_1 \cap W_2).$$

**Problem 8.33.** *Let $V_1$, $V_2$, and $V_3$ be vector spaces over a field $\mathbb{F}$, and let $\phi_{12} : V_1 \to V_2$ and $\phi_{23} : V_2 \to V_3$ be linear transformations. Suppose in addition that $\phi_{23} \circ \phi_{12} : V_1 \to V_3$ is the zero transformation. Prove that*

$$\dim_{\mathbb{F}} (V_1) - \dim_{\mathbb{F}} (V_2) = \dim_{\mathbb{F}} (\ker \phi_{12}) - \dim_{\mathbb{F}} \left( \frac{\ker \phi_{23}}{\mathrm{Im}\phi_{12}} \right) - \dim_{\mathbb{F}} (\mathrm{Im}\phi_{23}).$$

## 9. Eigenvalues and Eigenvectors

**Problem 9.1.** *Show that* $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ *is an eigenvector of the matrix* $\begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{3}{2} & -1 \end{pmatrix}$.

**Problem 9.2.** *Show that* $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *is not an eigenvector of the matrix* $\begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{3}{2} & -1 \end{pmatrix}$.

**Problem 9.3.** *Show that* $\begin{pmatrix} 1-i \\ 1 \end{pmatrix}$ *is an eigenvector of the matrix* $\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$.

**Problem 9.4.** *Show that* $\begin{pmatrix} -\frac{1}{2} - \frac{5}{2}i \\ -1-i \\ 1 \end{pmatrix}$ *is an eigenvector of the matrix* $\begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{pmatrix}$.

**Problem 9.5.** *Let $C^1(\mathbb{R})$ be the vector space of functions on $\mathbb{R}$ whose derivatives are continuous. Let $C^0(\mathbb{R})$ be the vector space of functions on $\mathbb{R}$ that are continuous. Let $L : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ be the function defined by $L(g) = g' - g$.*

(a) *Show that $L$ is a linear transformation.*
(b) *Show that the function $h$ defined by $h(x) = e^{7x}$ is an eigenfunction of $L$. Find the corresponding eigenvalue.*

**Problem 9.6.** *For each matrix below, find all the eigenvalues. Then find the eigenspace associated to each eigenvalue, and give an example of an eigenvector corresponding to that eigenvalue.*

(a) $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$
(b) $\begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$
(c) $\begin{pmatrix} 2 & -1 \\ 0 & -3 \end{pmatrix}$
(d) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$
(e) $\begin{pmatrix} 7 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$
(f) $\begin{pmatrix} 5 & -4 \\ 1 & 1 \end{pmatrix}$
(g) $\begin{pmatrix} 2 & 0 \\ 0 & -5 \end{pmatrix}$
(h) $\begin{pmatrix} 19 & 4 & -12 \\ 0 & 3 & 0 \\ 16 & 4 & -9 \end{pmatrix}$

**Problem 9.7.** *Show that the eigenvalues of the matrix $\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$ are $a$, $d$, and $f$.*

**Problem 9.8.** *Find a $2 \times 2$ matrix $M$ whose transformation $T_M$ rotates points of $\mathbb{R}^2$ by an angle of $-60$ degrees.*

**Problem 9.9.** *Find a $2 \times 2$ matrix $M$ whose transformation $T_M$ multiplies points along the vector $(2, -3)$ by $7$ and maps $(1, 0)$ to itself.*

**Problem 9.10.** *Find a $2 \times 2$ matrix $A$ whose transformation $T_A$ rotates points of $\mathbb{R}^2$ by 45 degrees and then increases the distance between the origin and the point by a factor of 8.*

**Problem 9.11.** *Find a $2 \times 2$ matrix $A$ whose transformation $T_A$ stretches the vector $(3, -1)$ by a factor of $2$ and stretches the vector $(2, 7)$ by a factor of $3$.*

**Problem 9.12.** *Find a $2 \times 2$ matrix whose transformation $T_A$ stretches the $x$-axis direction by a factor of $5$, then reflects across the $x$-axis, then rotates by $+60$ degrees.*

**Problem 9.13.** *Suppose that $v \in \mathbb{R}^n$ is an eigenvector for the $n \times n$ matrix $A$ (with eigenvalue $\lambda$) and is also an eigenvector for the $n \times n$ matrix $B$, but for a different eigenvalue $\mu$. Prove that $v$ is also an eigenvector for the matrix $A^2 - 2AB + 3I$.*

**Problem 9.14.** *Assume that $A$ and $B$ are $n \times n$ matrices.*

(a) Solve for the matrix $A$ in the equation below, and simplify. State any assumptions that are needed.
$BAB + B = 3B^2$

(b) Based on your equation above, suppose that $v$ is an eigenvector for $B$ corresponding to the eigenvalue $\frac{2}{3}$. Prove that $v$ is also an eigenvector for $A$, and find the corresponding eigenvalue.

**Problem 9.15.** *Find a* $2 \times 2$ *matrix* $A$ *whose eigenvalues are* $5$ *and* $-1$*, corresponding to eigenvectors* $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ *and* $\begin{pmatrix} 7 \\ 11 \end{pmatrix}$*, respectively.*

**Problem 9.16.** *Find a* $2 \times 2$ *matrix* $B$ *whose associated transformation* $T_B : \mathbb{R}^2 \to \mathbb{R}^2$ *rotates points counterlockwise by* $135$ *degrees.*

**Problem 9.17.** *Prove that if* $B$ *and* $C$ *are two* $n \times n$ *matrices and* $BC + 3CB = 0$*, then either* $B$ *or* $C$ *must have determinant zero.*

**Problem 9.18.** *If* $X$ *is a square matrix, prove that* $X^T$ *has the same eigenvalues as the matrix* $X$*, including algebraic multiplicities.*

**Problem 9.19.** *Let* $M$ *and* $A$ *be invertible* $n \times n$ *matrices such that* $MA = -AM$*. For each eigenvalue* $\lambda$ *of* $M$*, let* $G_\lambda(M)$ *denote the generalized eigenspace associated to* $\lambda$*. Prove that the transformation* $T_A$ *maps* $G_\lambda(M)$ *to* $G_{-\lambda}(M)$*, so that in particular* $-\lambda$ *is an eigenvalue of* $M$ *if* $\lambda$ *is.*

**Problem 9.20.** *Consider the two polynomials* $(x+1)$ *and* $x^2 - 3x - 4$*.*

(a) *Find their greatest common divisor.*

(b) *Find polynomials* $a(x)$ *and* $b(x)$ *such that* $a(x)(x+1) + b(x)(x^2 - 3x - 4)$ *is the previous answer.*

**Problem 9.21.** *Consider the two polynomials* $x + 2$ *and* $x^2 - 3x - 4$*.*

(a) *Show they are relatively prime.*

(b) *Find polynomials* $a(x)$ *and* $b(x)$ *such that* $a(x)(x+2) + b(x)(x^2 - 3x - 4) = 1$*.*

**Problem 9.22.** *Prove that the greatest common divisor of two polynomials* $p(x)$ *and* $q(x)$ *is unique.*

**Problem 9.23.** *Show that the following subspaces* $U, V$ *satisfy* $U \oplus V = \mathbb{R}^3$*.*

(a) $U = \{0\}$, $V = \mathbb{R}^3$.

(b) $U = \{(x, x, 0) : x \in \mathbb{R}\}$, $V = \{(0, y, z) : y, z \in \mathbb{R}\}$.

(c) $U = \{(x, 2x, y) : x, y \in \mathbb{R}\}$, $V = \{(x, 3x, 2x) : x \in \mathbb{R}\}$.

**Problem 9.24.** *For each matrix below, find all of the generalized eigenspaces.*

(a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 9 & 0 \\ -1 & -5 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} -1 & 2 & -5 \\ 0 & -26 & 75 \\ 0 & -10 & 29 \end{pmatrix}$

**Problem 9.25.** *Find the characteristic and minimal polynomials of each matrix below.*

(a) $\begin{pmatrix} 1 & 4 \\ 8 & -1 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 9 & 63 \\ -8 & -19 & -112 \\ 1 & 2 & 11 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & 890 & 4525 \\ 0 & 2 & 43 \\ 0 & 0 & 4 \end{pmatrix}$

**Problem 9.26.** *Let $p(x)$ and $q(x)$ be the characteristic and minimal polynomials, respectively, of the square matrix $A$. What can you say about the characteristic and minimal polynomials of the matrix $A^2$?*

**Problem 9.27.** *Suppose that $A^2 = 4A$ for some square matrix $A$. Prove or disprove that there exists a (nonzero) generalized eigenvector of $A$ that is not an eigenvector of $A$.*

**Problem 9.28.** *Prove that if $\lambda$ is a generalized eigenvalue of the square matrix $A$, then $a\lambda + b$ is a generalized eigenvalue of the matrix $aA + bI$.*

**Problem 9.29.** *Suppose that a 2x2 matrix satisfies $M^2 = 2M - I$.*

(a) *Prove that $M$ is invertible.*
(b) *Prove or disprove that there necessarily exists a basis of $\mathbb{C}^2$ consisting of eigenvectors of $M$.*

**Problem 9.30.** *Put each of the matrices below into Jordan canonical form using the rank algorithm, and find the minimal and characteristic polynomials of the matrices.*

(a) $\begin{pmatrix} -6 & -1 \\ 4 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 1 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 18 & 9 & 1 & 0 \\ -4 & 0 & 0 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 1 & 0 & 0 \\ -5 & 3 & 1 & 0 \\ 22 & -7 & -3 & 0 \\ -5 & 4 & 1 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & -1 & -2 \\ -2 & 3 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} -15 & 36 & 0 \\ -8 & 19 & 0 \\ 6 & -12 & 3 \end{pmatrix}$

(f) $\begin{pmatrix} -2 & 1 & 5 & 4 \\ 0 & -2 & 6 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & -4 \end{pmatrix}$

**Problem 9.31.** *Find a matrix $P$ such that $P^{-1}MP$ is in Jordan form, if $M$ is the matrix below.*

(a) $\begin{pmatrix} -6 & -1 \\ 4 & -2 \end{pmatrix}$

(b) $\begin{pmatrix} 3 & 1 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 18 & 9 & 1 & 0 \\ -4 & 0 & 0 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 1 & 0 & 0 \\ -5 & 3 & 1 & 0 \\ 22 & -7 & -3 & 0 \\ -5 & 4 & 1 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 3 & -1 & -2 \\ -2 & 3 & 3 \\ 1 & -1 & 0 \end{pmatrix}$

(e) $\begin{pmatrix} -15 & 36 & 0 \\ -8 & 19 & 0 \\ 6 & -12 & 3 \end{pmatrix}$

(f) $\begin{pmatrix} -2 & 1 & 5 & 4 \\ 0 & -2 & 6 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & -4 \end{pmatrix}$

**Problem 9.32.** *Let $A$ and $B$ be both diagonalizable matrices, and suppose that $AB = BA$.*

(a) *Prove that there exists an invertible matrix $M$ such that $M^{-1}AM$ and $M^{-1}BM$ are both diagonal matrices.*
(b) *Prove that $B$ is a polynomial in $A$.*

**Problem 9.33.** *Suppose that $A$ is a $7 \times 7$ matrix.*

(a) *Prove the statement: If $(A - 4I)$ and $(A - 4I)^2$ have different ranks, then there does not exist an invertible matrix $P$ such that $P^{-1}AP$ is diagonal.*
(b) *Is the converse to the previous statement true?*
(c) *Is the converse to (1) true, if it is assumed from the beginning that 4 is an eigenvalue of $A$?*

**Problem 9.34.** *Find a matrix $3 \times 3$ matrix $B$ such that it maps $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ to $\begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Is the answer to this problem unique?*

**Problem 9.35.** *Find a matrix $3 \times 3$ matrix $B$ such that it has an eigenvalue 2 and it maps $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ to $\begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix}$. Is the answer to this problem unique?*

**Problem 9.36.** *Prove or disprove that if $c$ is an eigenvalue of the matrix $A$ and* $\det(A) = 5$, *then $\frac{1}{c}$ is an eigenvalue of $A^{-1}$.*

**Problem 9.37.** *Prove or disprove that if $A$ is an invertible $n \times n$ matrix, then there exists a number $R > 0$ such that $(A + xI)$ is invertible for all $x > R$.*

**Problem 9.38.** *Let $M$ be an $5 \times 5$ matrix such that $M^2 + M^3 = 0$. Find all possible Jordan forms of the matrix $M$.*

**Problem 9.39.** *Let $B$ be an $n \times n$ matrix that is not diagonal, and suppose that $(B - 4I)^4 = 0$. Prove that for every invertible matrix $C$, $CBC^{-1}$ is not diagonal.*

**Problem 9.40.** *Let $B$ be an $n \times n$ matrix such that $(B - xI)(B - yI) = 0$.*

(a) *Prove or disprove that if $x \neq y$, then $B = xI$ or $B = yI$.*
(b) *Prove or disprove that if $x \neq y$, then $B$ is diagonalizable.*
(c) *Prove or disprove that if $x = y$, then $B$ is diagonalizable.*

**Problem 9.41.** *Let $M$ be an $n \times n$ matrix such that $M^3 = I$.*

(a) *Is it always true that $M$ is the identity matrix?*
(b) *Prove that there exists a basis of $\mathbb{C}^n$ consisting of eigenvectors of $M$.*

**Problem 9.42.** *Suppose that a $5 \times 5$ matrix $A$ satisfies $A^3 = 0$.*

(a) *Find the possible Jordan forms (up to equivalence) of $A$.*
(b) *For each Jordan form, find the corresponding minimal polynomial of $A$.*
(c) *For each Jordan form, find the dimension of the zero eigenspace.*
(d) *For each Jordan form, find the rank of the matrix $A^j$ for $j = 1, 2, 3, 4, 5, 6$.*
(e) *How many conjugacy classes are there in the set $\{B \in M_5(\mathbb{C}) : B^3 = 0\}$? ($M_5(\mathbb{C})$ is the set of $5 \times 5$ matrices with complex number entries.)*

**Problem 9.43.** (a) *Give an example of an* $8 \times 8$ *matrix* $M$ *such that* $(M + 3I)^3 = 0$ *and* $(M + 3I)^2 \neq 0$*, where* $0$ *denotes the zero matrix.*
(b) *Must every such* $M$ *be invertible?*

**Problem 9.44.** *Let* $X$ *be a* $3 \times 3$ *matrix such that* $\{v : (X - 2I)\, v = 0\}$ *has dimension* $2$ *and* $\left\{v : (X - 2I)^3\, v = 0\right\}$ *also has dimension* $2$.
(a) *Prove that there exists an eigenvalue of* $X$ *that is not* $2$.
(b) *Prove that* $X$ *is diagonalizable.*

**Problem 9.45.** *Let* $A$ *be a real* $6 \times 6$ *matrix of rank* $4$. *What are the possible ranks of* $A^2$*?*

**Problem 9.46.** *For an* $n \times n$ *matrix* $A$ *and eigenvalue* $\lambda_0$*, prove using basic facts that the dimension for the eigenspace for* $\lambda_0$ *is at most its multiplicity as a root of the characteristic polynomial.*

**Problem 9.47.** *Define the* ***rank*** *of a matrix over a field* $\mathbb{F}$. *Prove that if* $A$ *and* $B$ *are matrices over* $\mathbb{F}$ *such that* $AB$ *exists, then the rank of* $AB$ *over* $\mathbb{F}$ *is at most the minimum of the ranks of* $A$ *and of* $B$ *over* $\mathbb{F}$.

**Problem 9.48.** *Suppose that* $AB + BA$ *is a matrix with all entries* $0$. *Prove that:*
(a) $A$ *and* $B$ *are square matrices of the same size.*
(b) *If* $v$ *is an eigenvector of* $A$ *corresponding to eigenvalue* $\alpha$*, then* $Bv$ *is either* $0$ *or an eigenvector of* $A$.
(c) *If* $A$ *and* $B$ *are* $n \times n$ *matrices with* $n$ *odd, prove that at least one of the two matrices is singular.*

**Problem 9.49.** *Suppose that* $\det(A + xB) = x^5 + 10x + 5$ *for* $5 \times 5$ *matrices* $A$ *and* $B$ *with complex coefficients and all* $x \in \mathbb{C}$. *Prove that* $B$ *is an invertible matrix.*

**Problem 9.50.** *Suppose that* $\det(xC - D) = x^7 + 10x + 30$ *for* $7 \times 7$ *matrices* $C$ *and* $D$ *with complex entries and any* $x \in \mathbb{C}$. *Find* $\det C$, $\det D$, *and* $\mathrm{Tr}(C^{-1}D)$. *Furthermore, prove that* $C^{-1}D$ *is diagonalizable.*

## 10. Diagonalization and Inner Products

**Problem 10.1.** *Determine whether or not each matrix* $A$ *below is diagonalizable. If so, find a matrix* $P$ *such that* $P^{-1}AP$ *is diagonal. If not, explain why not.*
(a) $A = \begin{pmatrix} -17 & 36 \\ -6 & 13 \end{pmatrix}$
(b) $A = \begin{pmatrix} 13 & 25 \\ -1 & 3 \end{pmatrix}$
(c) $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$
(d) $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$
(e) $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 10 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

**Problem 10.2.** *Prove that if* $M$ *is any* $n \times k$ *real matrix then* $M^T M$ *is a symmetric matrix, and all of its eigenvalues are nonnegative.*

**Problem 10.3.** *Prove that if $B$ is any $n \times k$ matrix, then the matrix $BB^*$ is Hermitian symmetric, and all of its eigenvalues are nonnegative real numbers.*

**Problem 10.4.** *Prove that if $C$ is a square matrix, then $\frac{1}{2}(C + C^*)$ is Hermitian symmetric.*

**Problem 10.5.** *Prove the equation $2\mathrm{Re}(\langle v, w\rangle) = \|v+w\|^2 - \|v\|^2 - \|w\|^2$ for any $v, w \in \mathbb{C}^n$.*

**Problem 10.6.** *Suppose that $\lambda$ is an eigenvalue for the invertible matrix $M$. Prove that $\frac{1}{\lambda}$ is an eigenvalue for $M^{-1}$.*

**Problem 10.7.** *Prove that if $M$ is an $n \times n$ matrix, then $\det(M)$ is the product of all the eigenvalues of $M$, repeated according their multiplicities.*

**Problem 10.8.** *Suppose that $A$ is a $5 \times 5$ matrix, and $\det(A - \lambda I) = -(\lambda - 2)^3 (\lambda - 1)(\lambda - 4)$.*

(a) *List all the eigenvalues of $A$ and their multiplicities.*
(b) *Is it necessarily true that $A$ is invertible?*
(c) *What are the possible dimensions of the eigenspace corresponding to the eigenvalue 2?*
(d) *What are the possible dimensions of the eigenspace corresponding to the eigenvalue 1?*
(e) *What are the possible numbers that could be the rank of $A$?*
(f) *What are the possible numbers that could be the nullity of $A$?*
(g) *Do any of the previous answers change if you know that $A$ is Hermitian symmetric?*

**Problem 10.9.** *Suppose that $A$ is a $5 \times 5$ matrix, and $\det(A - \lambda I) = -\lambda^2 (\lambda - 2)(\lambda - 1)^2$*

(a) *List all the eigenvalues of $A$ and their multiplicities.*
(b) *Is it necessarily true that $A$ is invertible?*
(c) *What are the possible dimensions of the eigenspace corresponding to the eigenvalue 2?*
(d) *What are the possible dimensions of the eigenspace corresponding to the eigenvalue 1?*
(e) *What are the possible numbers that could be the rank of $A$?*
(f) *What are the possible numbers that could be the nullity of $A$?*
(g) *Do any of the previous answers change if you know that $A$ is Hermitian symmetric?*

**Problem 10.10.** *Suppose that the matrix $n \times n$ matrix $P$ has $n$ different eigenvalues. Prove that $P$ is diagonalizable.*

**Problem 10.11.** *Prove that if $B$ is a Hermitian symmetric $n \times n$ matrix with $\mathrm{rk}(B) < n$, then the nullity of $B$ is the same as the multiplicity of the zero eigenvalue of $B$.*

**Problem 10.12.** *Suppose that the $n \times n$ matrix $A$ is diagonalizable and that the only eigenvalues of $A$ are 2 and $-2$. Prove that $A^2 = 4I$.*

**Problem 10.13.** *Find an orthogonal matrix $U$ such that $U^{-1}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} U$ is diagonal, or prove that no such matrix exists.*

**Problem 10.14.** *Find an orthonormal basis of $\mathbb{R}^2$ consisting of eigenvectors of the matrix* $\begin{pmatrix} -1 & -12 \\ -12 & 6 \end{pmatrix}$.

**Problem 10.15.** *Find an orthonormal basis of $\mathbb{R}^2$ consisting of eigenvectors of the matrix* $\begin{pmatrix} 1 & -2\sqrt{3} \\ -2\sqrt{3} & 5 \end{pmatrix}$.

**Problem 10.16.** *Find an orthonormal basis of* $\mathbb{C}^2$ *consisting of eigenvectors of the matrix* $\begin{pmatrix} 3 & i \\ -i & 3 \end{pmatrix}$.

**Problem 10.17.** *Find an orthonormal basis of* $\mathbb{C}^2$ *consisting of eigenvectors of the matrix* $\begin{pmatrix} -1 & -12 \\ -12 & 6 \end{pmatrix}$.

**Problem 10.18.** *Show that* $\{(1,2),(-2,1)\}$ *is an orthogonal basis of* $\mathbb{R}^2$.

**Problem 10.19.** *Consider the plane* $P = \operatorname{span}\{(1,2,2),(2,2,1)\} \subseteq \mathbb{R}^3$. *Find an orthonormal basis of* $P$.

**Problem 10.20.** *Find an orthonormal basis for the plane spanned by* $(1,2,0,1)$ *and* $(0,-2,-1,1)$ *in* $\mathbb{R}^4$.

**Problem 10.21.** *Prove directly that if* $a,b,c$ *are linearly independent vectors in* $\mathbb{C}^3$, *then* $\left\{a,\widetilde{b},\widetilde{c}\right\}$ *is an orthogonal basis of* $\mathbb{C}^3$, *where*

$$\widetilde{b} = b - \frac{\langle b,a\rangle}{\langle a,a\rangle}a \ ; \ \widetilde{c} = c - \frac{\langle c,a\rangle}{\langle a,a\rangle}a - \frac{\left\langle c,\widetilde{b}\right\rangle}{\left\langle \widetilde{b},\widetilde{b}\right\rangle}\widetilde{b}.$$

**Problem 10.22.** *Prove that if* $\{v,w,z\}$ *is an orthonormal basis of* $\mathbb{C}^3$, *then for any vector* $p \in \mathbb{C}^3$,

$$p = \langle p,v\rangle v + \langle p,w\rangle w + \langle p,z\rangle z.$$

**Problem 10.23.** *Show that* $v = \left(\frac{2}{5}, -\frac{2\sqrt{3}}{5}, \frac{3}{5}\right)$ *is a unit vector. Find two other vectors* $a,b \in \mathbb{R}^3$ *so that* $\{v,a,b\}$ *is an orthonormal basis of* $\mathbb{R}^3$.

**Problem 10.24.** *Construct a matrix* $A$ *such that the linear transformation* $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ *stretches by a factor* $3$ *in the direction of the vector* $(-2,5)$ *and satisfies* $T_A(v) = v$ *for all vectors* $v$ *perpendicular to* $(-2,5)$.

**Problem 10.25.** *Prove that if* $M$ *is a symmetric real matrix with all eigenvalues* $1$, *then* $M$ *must be the identity matrix. Prove that the previous statement is false if the word "symmetric" is removed.*

**Problem 10.26.** *Prove the following are equivalent for an* $n \times n$ *real matrix* $A$.

(a) *The columns of* $A$ *form an orthonormal basis for* $\mathbb{R}^n$.
(b) *The rows of* $A$ *form an orthonormal basis for* $\mathbb{R}^n$.
(c) *For every* $x \in \mathbb{R}^n$, $||Ax|| = ||x||$.
(d) *For every* $x,y \in \mathbb{R}^n$, $Ax \cdot Ay = x \cdot y$.
(e) $A^TA = I$.

**Problem 10.27.** *True or False: For any two* $n \times n$ *complex matrices* $A$ *and* $B$,

(a) $A+B$ *is nonsingular if* $A$ *and* $B$ *are nonsingular.*
(b) $A+B$ *is nonsingular if* $A$ *and* $B$ *are real symmetric matrices and all of their eigenvalues are strictly positive.*
(c) $A+B$ *is nonsingular if all the eigenvalues of* $A+A^*$ *and* $B+B^*$ *are strictly positive.*

**Problem 10.28.** *Prove or disprove that there exists a real symmetric matrix* $M$ *such that*

(a) $M^3 = \begin{pmatrix} 1 & 3 \\ -6 & 5 \end{pmatrix}$

(b) $M^2 = \begin{pmatrix} 41 & -6 \\ -6 & 37 \end{pmatrix}$

(c) $M^3 = \begin{pmatrix} 41 & 6 \\ 6 & -39 \end{pmatrix}$

(d) $M^2 = \begin{pmatrix} 41 & 6 \\ 6 & -39 \end{pmatrix}$

(e) $M^3 = \begin{pmatrix} 1 & -1 & 3 & 2 \\ -1 & 4 & -1 & 7 \\ 3 & -1 & 2 & -1 \\ 2 & 7 & -1 & 8 \end{pmatrix}$

(f) $M^2 = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 4 & -1 \\ 2 & -1 & 2 \end{pmatrix}$

**Problem 10.29.** *Prove that if $A$ is a real symmetric $n \times n$ matrix, then*

$$\inf_{v \in \mathbb{R}^n \setminus \{0\}} \frac{\langle Av, v \rangle}{\langle v, v \rangle}$$

*is necessarily an eigenvalue of $A$. Is the statement true, if $A$ is not required to be symmetric?*

**Problem 10.30.** *Suppose that a $7 \times 7$ complex matrix $M$ satisfies $(M - I)(M^* - 2I) = 0$. Prove that $M^* = M$, and find all possible sets of eigenvalues of $M$.*

**Problem 10.31.** *Let $P_2 = P_2(\mathbb{R})$ denote the real vector space of real polynomials of degree at most 2.*

(a) *Prove that $\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$ is an inner product on $P_2$.*

(b) *Show that there is a unique $q \in P_2$ such that*

$$\int_{-1}^{1} p(x)\ \cos(\pi x)\, dx = \langle p, q \rangle$$

*for all $p$ and find it.*

**Problem 10.32.** *Define an inner product on the space of real polynomials by*

$$\langle f, g \rangle = \int_0^\infty f(x) g(x) e^{-x}\, dx.$$

(a) *Find an orthonormal basis for the subspace spanned by $1$, $x$, and $x^2$.*

(b) *Calculate the projection of $x^3$ onto this subspace.*

**Problem 10.33.** *Suppose that* $\det(A + xB) = x^5 + 10x + 5$ *for $5 \times 5$ matrices $A$ and $B$ with complex entries and all $x \in \mathbb{C}$. Prove that $B$ is an invertible matrix.*

**Problem 10.34.** *Let $A$ be an $n \times m$ real matrix of rank $n$. Prove that $P(x) = A \left(A^T A\right)^{-1} A^T x$ is the orthogonal projection of $x \in \mathbb{R}^n$ to the column space of $A$.*

**Problem 10.35.** *Let $M$ be a real $6 \times 6$ matrix of rank 4. What are the possible ranks of $M^2$?*

**Problem 10.36.** *Prove that if $\{b_1, ..., b_k\}$ is an orthogonal set of vectors in an inner product space $(V, \langle \bullet, \bullet \rangle)$, then $\{b_1, ..., b_k\}$ is linearly independent.*

**Problem 10.37.** *Find an orthonormal basis for the plane $3z + 4x - y = 0$ in $\mathbb{R}^3$.*

**Problem 10.38.** *Prove that if $M$ is an $n \times n$ orthogonal matrix, then*

(a) *its columns form an orthonormal basis of $\mathbb{R}^n$.*
(b) *its rows form an orthonormal basis of $\mathbb{R}^n$.*

**Problem 10.39.** *Starting from the set of vectors $\{(1,1,1),(1,0,0),(0,1,0)\}$, use the Gramm-Schmidt procedure to construct an orthonormal basis for $\mathbb{R}^3$.*

**Problem 10.40.** *Consider the vector space of polynomials $P$ with real coefficients and the real inner product*

$$\langle p, q\rangle := \int_{-\infty}^{\infty} p(x)\, q(x)\, e^{-x^2} dx.$$

(a) *Prove that $\langle \bullet, \bullet \rangle$ is a real inner product on $P$.*
(b) *Prove that $p(x) = 1$ and $q(x) = x^5$ are orthogonal.*
(c) *Let $S = \{1, x, x^2, x^3\}$. Prove that $S$ is a linearly independent set.*
(d) *Let $F(k) = \int_{-\infty}^{\infty} x^{2k} e^{-x^2} dx$ for $k \in \mathbb{Z}_{\geq 0}$. Prove that*

$$F(k) = \frac{1 \cdot 3 \cdot 5 \cdot \ldots (2k-1)}{2^k}\sqrt{\pi}.$$

(e) *Use the Gramm-Schmidt procedure to find two polynomials $p_2$ and $p_3$ such that $\{1, x, p_2(x), p_3(x)\}$ is an orthogonal set of vectors that spans the same space as $S$ above.*
(f) *Change the previous orthogonal set of vectors to an orthonormal set of vectors.*

**Problem 10.41.** *Let $V$ be a real inner product space. Prove that $|\langle v, w\rangle| \leq \|v\| \; \|w\|$.*

**Problem 10.42.** *Let $V$ be a complex inner product space. Prove that $|\langle v, w\rangle| \leq \|v\| \; \|w\|$.*

**Problem 10.43.** *Prove or disprove that real antisymmetric matrices (real matrices $M$ such that $M^T = -M$) are normal.*

**Problem 10.44.** *Let $A$ be a skew-Hermitian matrix, so that $A^* = -A$.*

(a) *Prove that $iA$ is Hermitian symmetric.*
(b) *Prove that there exists an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of $A$.*
(c) *Prove that every eigenvalue of $A$ is pure imaginary*
(d) *Let $M$ be any matrix such that $D = U^* M U$ is diagonal for some unitary matrix $U$, and suppose that $D^* = -D$. Prove that $M$ is skew-Hermitian.*

**Problem 10.45.** *Let $P$ be an orthogonal projection matrix, so that $P^2 = P$ and $P^* = P$.*

(a) *Find all eigenvalues of $P$.*
(b) *Prove that $P$ is normal.*
(c) *If $P^* \neq P$, is it possible that $P$ is not normal?*

**Problem 10.46.** *Give an example of a diagonalizable matrix that is not normal.*

**Problem 10.47.** *Prove or disprove that the product of two normal matrices is normal.*

**Problem 10.48.** *Prove that each matrix below is normal, and find a unitary matrix that diagonalizes it.*

(a) $\begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$
(b) $\begin{pmatrix} 4 & 3 \\ 3 & -4 \end{pmatrix}$
(c) $\begin{pmatrix} 16-9i & 12+12i \\ 12+12i & 9-16i \end{pmatrix}$

**Problem 10.49.** *Let* $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. *Determine (concise) necessary and sufficient conditions on a field* $\mathbb{F}$ *so that* $A$ *is diagonalizable over* $\mathbb{F}$.

**Problem 10.50.** *Let* $M$ *be a unitary matrix (ie* $M$ *is square and* $\overline{M^T}M = I$ *). Prove from basic definitions that*

(a) *All eigenvalues of* $M$ *have absolute value* 1.
(b) *Eigenvectors of* $M$ *corresponding to different eigenvalues are (complex) orthogonal.*

**Problem 10.51.** *Let* $V$ *be an inner product space. Let* $W$ *be a subspace of* $V$*, and let* $W^\perp$ *denote its orthogonal complement.*

(a) *For* $V$ *finite dimensional, prove* $\left(W^\perp\right)^\perp = W$.
(b) *For* $V = \mathbb{R}[x]$*, the vector space of polynomials with real coefficients, we define the inner product*

$$\langle p, q\rangle = \int_{-1}^{1} p(x)\, q(x)\; dx.$$

*Let* $W$ *be the subspace with basis* $\{x^2, x^4, x^6, ...\}$ *. Find bases for* $W^\perp$ *and* $\left(W^\perp\right)^\perp$*, and show that* $W \subsetneq \left(W^\perp\right)^\perp$.

**Problem 10.52.** *Let* $A = \begin{pmatrix} -1 & 4 & 5 \\ -4 & 10 & 8 \\ 5 & -12 & -9 \end{pmatrix}$.

(a) *Determine a matrix* $P$ *and a diagonal matrix* $D$ *such that* $D = P^{-1}AP$.
(b) *Compute* $A^{2015}$.

**Problem 10.53.** *Let* $B$ *be a real matrix that is diagonalizable over* $\mathbb{R}$*. Prove that* $B$ *is symmetric if and only if eigenvectors corresponding to distinct eigenvalues are orthogonal.*

**Problem 10.54.** *Let* $A$ *be a complex* $7 \times 7$ *matrix such that is both Hermitian symmetric and unitary.*

(a) *Prove that* $\mathrm{tr}A$ *is an integer.*
(b) *Assuming that* $A$ *is not a scalar multiple of the identity matrix, prove that there exists a* 2*-dimensional (complex) subspace* $V$ *of* $\mathbb{C}^7$ *such that*
   (i) *multiplication* $m_A$ *by* $A$ *is a linear transformation* $m_A : V \to V$*, and*
   (ii) *there exists an orthonormal basis* $\{v_1, v_2\}$ *of* $V$ *such that* $m_A(v_1) = v_2$ *and* $m_A(v_2) = v_1$.

## 11. Applications of diagonalization, inner products, and Jordan form

**Problem 11.1.** *Let* $S$ *be the subspace* $\{f \in L^2(S^1) : f(-x) = -f(x)\} \subset L^2(S^1)$.

(a) *Prove that* $\left\{\frac{1}{\sqrt{\pi}} \sin(jx)\right\}_{j=1}^{\infty}$ *is an orthonormal basis for* $S$.
(b) *Show that the function* $g$ *defined by* $g(\theta) = -2\theta$ *for* $\theta \in (-\pi, \pi)$ *is in* $S$*. Express* $g(\theta)$ *in terms of the basis in (a).*
(c) *What equation does Parseval's equality for* $g$ *yield?*

**Problem 11.2.** *Let* $V$ *be the subspace of* $\mathbb{R}^n$ *that is the columnspace of the* $n \times k$ *matrix* $M$*. Suppose rank*$(M) = k$.

(a) *Prove that* $M^TM$ *is invertible.*

(b) *Prove that the orthogonal projection $P$ from $\mathbb{R}^n$ to $V$ satisfies*

$$P(x) = M\left(M^T M\right)^{-1} M^T x.$$

**Problem 11.3.** *Let $S$ be a subspace of $\mathbb{R}^n$ of dimension $k$. The orthogonal projection $P_S : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $P_S(x) = x$ for all $x \in S$ and $P_S(y) = 0$ for all vectors $y$ that are perpendicular to all vectors in $S$.*

(a) *Prove that $P_S \circ P_S = P_S$.*
(b) *Prove that $P_S^* = P_S$, where $*$ denotes the adjoint.*
(c) *Find the eigenvalues of $P_S$ and their multiplicities.*

**Problem 11.4.** *Consider the inner product $\langle f, g\rangle = \int_0^1 x f(x) g(x)\, dx$ on the space $C[0,1]$ of continuous functions on the interval $[0,1]$. Find an orthonormal basis for the subspace spanned by the functions $x$ and $x^2$. Find the projection of the function $h(x) = 1$ onto this subspace.*