Homework # 8 (Solutions).

1 (a) Let $A_n = (3 - 1/n, 3 + 1/n)$, then $\bigcap_{n=1}^{\infty} A_n = \{3\}$.

(b) Let $A_n = [3 - 1/n, 4 + 1/n]$, then $\bigcap_{n=1}^{\infty} A_n = [3,4]$.

(c) Let $A_n = (3 - 1/n, 4 + \frac{1}{n})$, then $\bigcap_{n=1}^{\infty} A_n = (3,4)$

2. From theorem 1.4.3 we know that there exists $r \in \mathbb{Q}$ s.t. $a < r < b$. We can use the Archimedian principle to choose $m \in \mathbb{N}$ s.t. $1/m < b - r$.
Then the numbers $C_k = r + \frac{1}{m+k}$, $k = 0, 1, 2, \ldots$ are all rational, distinct, and satisfy
$a < r < C_k < b \quad \forall\, k = 0, 1, 2, \ldots$

3. Let $A = \{ r \in \mathbb{Q} \mid r < a \}$, then a is an upper bound of A. To show that $a = \sup A$ we apply lemma 1.3.8. Given $\varepsilon > 0$, from theorem 1.4.3 there exists $s \in \mathbb{Q}$ s.t. $a - \varepsilon < s < a$. Since $s \in A$, by lemma 1.3.8 $\sup A = a$.

4. • $\bigcap_{n=1}^{\infty} A_n = \{0\}$. Observe that $0 \in A_n \ \forall n$, so

$0 \in \bigcap_{n=1}^{\infty} A_n$. Let $x \in \bigcap_{n=1}^{\infty} A_n$, then $x \in [0,1] \Rightarrow x \geq 0$.

If $x > 0$ then by the Archimedian principle $\exists\, m \in \mathbb{N}$, s.t. $\frac{1}{m} < x \Rightarrow x \notin A_m \Rightarrow x \notin \bigcap_{n=1}^{\infty} A_n$. Thus $x = 0$ is the only elt. in the intersection.

• Let $x \in \bigcap_{n=1}^{\infty} B_n$, then by the Archimedian principle $\exists\, m \in \mathbb{N}$ s.t. $x < m \Rightarrow x \notin B_m \Rightarrow x \notin \bigcap_{n=1}^{\infty} B_m \Rightarrow \bigcap_{n=1}^{\infty} B_m = \emptyset$.