# Homework #7 (Solutions)

1. ($\Rightarrow$) Assume that $\ell = \inf A$, then $\ell$ is the greatest lower bound $\Rightarrow$ $\forall \varepsilon > 0$, $\ell + \varepsilon$ is not a lower bound $\Rightarrow$ $\exists\, a \in A$ s.t. $\ell + \varepsilon > a$.

($\Leftarrow$) Assume that $\ell$ is a lower bound and $\forall \varepsilon > 0$ $\exists\, a \in A$ s.t. $\ell + \varepsilon > 0$, and yet $\ell \neq \inf A = \ell_1$ then $\ell_1 > \ell$. Choose $\varepsilon = \frac{\ell_1 - \ell}{2}$ and $a \in A$ such that $\ell + \varepsilon > a$. Then $\ell_1 > \ell + \varepsilon = \frac{\ell_1 + \ell}{2} > a$. This contradicts that $\ell_1$ is a lower bound of $A$.
$\Rightarrow$ $\ell = \inf A$.

2. We will show that $\sup A = 1$.
Observe that
$$\frac{n+1}{n+2} < \frac{n+2}{n+2} = 1,$$ so $s = 1$ is an upper bound of $A$.
We will apply lemma 1.3.7: given $\varepsilon > 0$ choose $n$ so that $\frac{1}{n+2} < \varepsilon$ (or $n > \frac{1}{\varepsilon} - 2$), and let $a = \frac{n+1}{n+2}$.
Then $\ell - \varepsilon < 1 - \frac{1}{n+2} = \frac{n+1}{n+2} = a$.

3. # 1.3.11 a) True. Let $S_1 = \sup A$ and $S_2 = \sup B$. Since $S_2$ is an upper bound of $B$ and $A \subseteq B$ $\Rightarrow$ $\forall a \in A$, $a \leq S_2$ $\Rightarrow$ $S_2$ is an upper bound of $A$. Then $S_1 = \sup A \leq S_2 = \sup B$ by definition of supremum.

b) True. Let $c = \frac{\sup A + \inf B}{2}$. Then $\forall a \in A$, $b \in B$,
$$a \leq \sup A < c < \inf B \leq b.$$

c) False. Let $A = (0,1)$, $B = (1,2)$, $c = 1$. Then $\forall a \in A$, $b \in B$, $a < c < b$. Yet $\sup A = 1 = \inf B$.