Homework # 6 (Solutions)

1. Let $A \subset \mathbb{R}$ be a set. A lower bound l of A is the infimum of A if and only if $\forall \varepsilon > 0$ $\exists a \in A$ s.t. $a < l + \varepsilon$.

2. Let $l_1 = \inf A$ and $l_2 = \inf B$. We need to show that $\inf(A+B) = l_1 + l_2$.
Observe that since l_1 and l_2 are lower bounds for A and B, $\forall a \in A, b \in B$, $a+b \geq l_1 + l_2$, so $l_1 + l_2$ is a lower bound for $A+B$.
To see that $l_1 + l_2 = \sup(A+B)$ we will verify criterion in 1:
Given $\varepsilon > 0$, since $l_1 = \inf A$ and $l_2 = \inf B$, $\exists a \in A$ s.t. $a < l_1 + \varepsilon/2$ and $\exists b \in B$ s.t. $b < l_2 + \varepsilon/2$.
Then $a + b < l_1 + l_2 + \varepsilon$, $a+b \in A+B \Rightarrow l_1 + l_2 = \inf(A+B)$.

1.3.3. (a) Since $\forall a \in A$, a is an upper bound of B, so B is bounded and since A is bounded, B is not empty. By the Completeness axiom, $s = \sup B$ exists. We now show that s is a lower bound of A. Suppose not, then $\exists a \in A$ s.t. $a < s$. Since any $a \in A$ is an upper bound of B, s is not $\sup B$, which is a contradiction.
s is the greatest lower bound of A since $\forall$ lower bound l of A, $l \in B \Rightarrow l \leq \sup B = s$. Thus A has an infimum and $\inf A = s = \sup B$.
(b) Given set $A \subset \mathbb{R}$ bounded below, consider set B as in (a). By Axiom of Completeness, there exists $\sup B$. By (a) $\inf A$ exists and is equal to $\sup B$