Solutions to homework #4.

1. We can rewrite the inequality as
$$||x|-|y|| \le |x-y|.$$
This inequality is equivalent to a double inequality
$$-|x-y| \le |x|-|y| \le |x-y|. \quad (1)$$
By the triangle inequality
$$|x| = |x-y+y| \le |x-y|+|y|, \text{ thus}$$
$$|x|-|y| \le |x-y|. \quad (2)$$
Similarly,
$$|y| = |y-x+x| \le |y-x|+|x|, \text{ that}$$
$$-|x-y| \le |x|-|y| \quad (3).$$

Combining (2) and (3) we obtain (1).

2. By the triangle inequality
$$|c| = |c-x+x| \le |c-x|+|x| = |x-c|+|x|$$
Since $|x-c| < \frac{|c|}{2}$, we get
$$|c| \le |x-c| + |x| < \frac{|c|}{2} + |x|$$
Thus, $|x| > |c| - \frac{|c|}{2} = \frac{|c|}{2}$.

3. $\boxed{1.2.10}$ a) This statement is false.
$(a<b) \Rightarrow (a<b+\varepsilon, \forall \varepsilon>0)$ part is true.
The "only if" part $(a<b+\varepsilon, \forall \varepsilon>0) \Rightarrow (a<b)$ is false: consider $a=b=0$, then the assumption $0<0+\varepsilon, \forall \varepsilon>0$ holds, yet the conclusion $0<0$ is false.

b) Says if $a < b + \varepsilon, \forall \varepsilon > 0$, then $a < b$, i.e.

$$(a < b + \varepsilon, \forall \varepsilon > 0) \Rightarrow (a < b).$$

We proved this to be false in (a).

c) This statement is true.

$(a \leq b) \Rightarrow (a < b + \varepsilon, \forall \varepsilon > 0)$:

Observe $a \leq b \Rightarrow a - b \leq 0 < \varepsilon \Rightarrow a < b + \varepsilon$.

$(a < b + \varepsilon, \forall \varepsilon > 0) \Rightarrow (a \leq b)$:

Proof by contradiction: Suppose $a > b$, then $a - b > 0$. Choosing $\varepsilon = \frac{a-b}{2} > 0$, observe that $a - b > \frac{a-b}{2} = \varepsilon \Rightarrow a > b + \varepsilon$, which contradicts the assumption.

1.2.11

(a) Negation: there exist real numbers a, b satisfying $a < b$ and such that $\forall n \in \mathbb{N}$ $a + \frac{1}{n} \geq b$.

The original statement is true

$\left(a < b \Rightarrow b - a > 0 \Rightarrow \exists n, \text{ s.t. } b - a > \frac{1}{n} > 0 \right)$

(b) For every real $x > 0$ there exists an $n \in \mathbb{N}$ s.t. $x \geq 1/n$. This statement is true (Archimedian principle).

(c) There exist two distinct real numbers such that there are no rational numbers between them. This statement is false, and the original statement is true