Hwk 33 (Solutions)

6.4.2 (a) True. Since $\sum_{n=1}^{\infty} g_n$ converges uniformly on A we can apply Cauchy criterion 6.4.4, to conclude that $\forall \varepsilon > 0 \ \exists N \geq \mathbb{N}$ such that $\forall n \geq N+1$ and $x \in A$, $|f_n(x)| < \varepsilon \Rightarrow$
$f_n \overset{A}{\rightrightarrows} 0$.

(b) True. Apply Cauchy Criterion 6.4.4.
Since $\sum_{n=1}^{\infty} g_n$ converges uniformly on A $\Leftrightarrow$
$\forall \varepsilon$, $\exists N \in \mathbb{N}$, s.t. $\forall m \geq N$, $k \in \mathbb{N}$, $\forall x \in A$
$|g_m(x) + g_{m+1}(x) + \ldots + g_{m+k}(x)| < \varepsilon$.
Since $0 \leq f_n(x) \leq g_n(x) \ \forall n \in \mathbb{N} \Rightarrow$
$|f_m(x) + \ldots + f_{m+k}(x)| = f_m(x) + \ldots + f_{m+k}(x)$
$\leq g_m(x) + \ldots + g_{m+k}(x) < \varepsilon$.
$\Rightarrow \sum_{n=1}^{\infty} f_n$ converges uniformly on A by the Cauchy criterion.

(c) False. Let $f_n(x) = \frac{(-1)^n}{n}$, then
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges on $\mathbb{R} = A$ (it is x-independent)
by alt. series test.
But if $\left| f_n \right| = \left| \frac{(-1)^n}{n} \right| \leq M_n$, then $M_n \geq 1/n \ \forall n \in \mathbb{N}$.
$M_n \geq \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} M_n \geq \sum_{n=1}^{\infty} \frac{1}{n}$, so $\sum_{n=1}^{\infty} M_n$ must diverge.

6.4.5 (a) Apply the M-series test:

On $[-1,1]$, $\left|\frac{x^n}{n^2}\right| \le \frac{1}{n^2} = M_n$.

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. by p-series test

Thus $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ conv. uniformly on $[-1,1]$.

(b) Let $x_0 \in (-1,1)$. We can choose $\ell > 0$ such that $-\ell < x_0 < \ell$. Then we apply the M-series test on $[-\ell,\ell]$. On this interval

$\left|\frac{x^n}{n}\right| \le \frac{\ell^n}{n} = M_n$ and $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{\ell^n}{n}$ converges

by the Ratio test:

$$\lim_{n\to\infty}\left|\frac{M_{n+1}}{M_n}\right| = \ell < 1$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n}$ converges uniformly on $[-\ell,\ell]$. Since all $\frac{x^n}{n}$ are continuous, $\sum_{n=1}^{\infty}\frac{x^n}{n}$ converges uniformly to a function that is continuous on $[-\ell,\ell]$ $\Rightarrow$ in particular, this function is continuous at x_0, $\ell < x_0 < \ell$.

6.4.7 We will apply Thm. 6.4.3.

Let $f_k(x) = \frac{\sin(kx)}{k^3}$ on $\mathbb{R}$, $f_k'(x) = \frac{\cos(kx)}{k^2}$ are continuous.

Observe that $|f_k'(x)| \le \frac{1}{k^2} = M_k$ on $\mathbb{R}$.

Since $\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges $\Rightarrow$

$\sum_{k=1}^{\infty} f_k'(x)$ converges uniformly on $\mathbb{R}$ to some continuous function g.

$\sum_{n=1}^{\infty} f_k(0) = 0 \Rightarrow$ by thm 6.4.3

$\sum_{n=1}^{\infty} f_n(x)$ converges to some differentiable

function $f(x)$ and $f'(x) = g(x)$, which we know must be continuous.

b) $\sum_{k=1}^{\infty} \left(\frac{\sin(kx)}{k^3}\right)'' = -\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$

Since the best estimate we can have is $\left|\frac{\sin(kx)}{k}\right| \leq \frac{1}{k}$ and the series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, we cannot determine convergence (even pointwise) of this series by the methods of this section.