Homework # 32 (Solutions)

\# 1. a) Given $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon}$. Then $\forall n \geq N$, $\forall x \in \mathbb{R}$

$$\left| \frac{\cos(n^2 x)}{n} - 0 \right| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus $\frac{\cos(n^2 x)}{n} \overset{\mathbb{R}}{\rightrightarrows} 0$.

b)

$$h_n'(x) = -n\sin(n^2 x)$$

$h_n'(x) = 0$, when $x = \pi k$, $k \in \mathbb{Z}$, so we have pointwise convergence when $x = \pi k$, $k \in \mathbb{Z}$.

\# 6.3.5 a) $\lim\limits_{n \to \infty} g_n(x) = \lim\limits_{n \to \infty} \left(\frac{x}{2} + \frac{x^2}{2n} \right) = \frac{x}{2}$, $\forall x \in \mathbb{R}$.

$$g'(x) = 1/2.$$

b) $g_n'(x) = 1/2 + \frac{x}{n} \overset{[-M, M]}{\rightrightarrows} h(x) = 1/2$.

This is true since $\forall \varepsilon > 0$ one can choose $N > M/\varepsilon$ then $\forall n \geq N$ and $\forall x \in [-M, M]$ one has $| g_n'(x) - h(x) | = \left| \frac{x}{n} \right| < \frac{M}{n} \leq \frac{M}{N} < \varepsilon$.

From thm. 6.3.3, $g_n \overset{[-M, M]}{\rightrightarrows} g$ and $g'(x) = h(x) = 1/2$ $\forall x \in [-M, M]$.

c) $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{(nx^2+1)}{2n+x}$

$$= \lim_{n\to\infty} \frac{\cancel{n}(x^2+\frac{1}{n})}{\cancel{n}(2+x/n)} = \frac{x^2}{2} = f(x) \quad \forall\, x \in \mathbb{R}.$$

$f'(x) = x$

Now $f_n'(x) = \frac{2nx(2n+x)-(nx^2+1)}{(2n+x)^2} = \frac{nx^2+4n^2x-1}{(2n+x)^2}$

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{\cancel{n^2}(4x+\frac{x^2}{n}-\frac{1}{n^2})}{\cancel{n^2}(4+\frac{4x}{n}+x^2/n^2)} = x = g(x)$$

We now prove that $\forall\, M>0$, $f_n'(x) \overset{[-M,M]}{\rightrightarrows} g(x)$.

Given $\varepsilon>0$, choose $N > \max\left\{4M, \frac{\varepsilon}{3M^2}, \sqrt{\frac{\varepsilon}{M^3+1}}\right\}$

$\forall\, n \geq N$ and $\forall\, x \in [-M,M]$

$$|f_n'(x) - g(x)| = \left| \frac{-\frac{3x^2}{n} - x^3/n^2 - \frac{1}{n^2}}{4+\frac{4x}{n}+x^2/n^2} \right|$$

$$\leq \frac{\frac{3M^2}{n}+\frac{M^3}{n^2}+\frac{1}{n^2}}{4-\frac{4M}{n}} \leq \frac{\varepsilon+\varepsilon}{3} < \varepsilon.$$

Thus $f_n' \overset{[-M,M]}{\rightrightarrows} g \Rightarrow f_n \overset{[-M,M]}{\rightrightarrows} f$ and $f' = g$.