Homework # 31

1 a) $\lim\limits_{n\to\infty} \dfrac{nx^2}{1+nx^4} = \lim\limits_{n\to\infty} \dfrac{\cancel{n}\cdot x^2}{\cancel{n}(\frac{1}{n}+x^4)} = \dfrac{x^2}{x^4} = \dfrac{1}{x^2} = f(x)$

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b) No, the convergence is not uniform.

Apply the negation of the def. of uniform convergence: Let $\varepsilon = 1/2$ and let $x_n = \dfrac{1}{n}$.

5 Observe that $|f_n(x) - f(x)| = \left|\dfrac{nx^2}{1+nx^4} - \dfrac{1}{x^2}\right|$

$= \dfrac{1}{x^2(1+nx^4)}$. $|f_n(x_n) - f(x_n)| = \dfrac{n^2}{1+\frac{1}{n^3}} > \dfrac{n^2}{2} \geq \dfrac{1}{2}$.

c) No, same reason as in b).

d) Yes, Given $\varepsilon > 0$, choose $N \geq \dfrac{1}{\varepsilon}$, then $\forall n \geq N$ and $\forall x \in (1, \infty)$,

5 $|f_n(x) - f(x)| = \dfrac{1}{x^2(1+nx^4)} \leq \dfrac{1}{1+n} < \dfrac{1}{N} < \varepsilon$.

2 a) Observe that $\lim\limits_{n\to\infty} x^n = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \\ +\infty, & x > 1 \end{cases}$

3 Then $\lim\limits_{n\to\infty} \dfrac{x^n - 1}{x^n + 1} = \begin{cases} -1, & 0 \leq x < 1 \\ 0, & x = 1 \\ 1, & x > 1. \end{cases} = g(x)$

b) If $g_n \overset{[0,+\infty)}{\rightrightarrows} g$, then g would be

5 continuous on $[0, +\infty)$. Since g is <u>not</u> continuous on $[0, +\infty)$, the convergence is not uniform by thm. 6.2.6

c) Consider $|g_n(x) - g(x)| = \begin{cases} \dfrac{2x^n}{x^n+1}, & 0 \le x < 1 \\ 0, & x = 1 \\ \dfrac{2}{x^n+1}, & x > 1. \end{cases}$

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• To prove $g_n \overset{[0,\frac{1}{2}]}{\rightrightarrows} g$, given $\varepsilon > 0$, choose $N \in \mathbb{N}$, so that $\frac{1}{2^{N-1}} < \varepsilon$ (i.e. $N > \frac{\ln(1/\varepsilon)}{\ln 2} + 1$).

Then $\forall n \geq N$ and $\forall x \in [0, \frac{1}{2}]$, $|g_n(x) - g(x)| = \frac{2x^n}{x^n+1} < 2x^n < \frac{1}{2^{n-1}}$

$\leq \frac{1}{2^{N-1}} < \varepsilon$.

To prove $g_n \overset{[2,+\infty)}{\rightrightarrows} g$, given $\varepsilon > 0$, choose the same $N \in \mathbb{N}$ as before. Then $\forall n \geq N$ and $\forall x \in [2, +\infty)$,

$|g_n(x) - g(x)| = \frac{2}{x^n+1} < \frac{2}{x^n} \leq \frac{2}{2^n} \leq \frac{1}{2^{N-1}} < \varepsilon$.

6.2.5 • Suppose $f_n \overset{A}{\rightrightarrows} f$, then given $\varepsilon > 0$, choose N s.t. $\forall n \geq N$, $\forall x \in A$

$|f_n(x) - f(x)| < \varepsilon/2$. Then $\forall m, n \geq N$, $\forall x \in A$

$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

$\Rightarrow$ (f_n) is uniformly Cauchy on A.

• Now assume that (f_n) is uniformly Cauchy on A.

$\Rightarrow$ $\forall c \in A$, sequence of numbers $f_n(c)$ is Cauchy.

By thm. 2.6.4. $\forall c \in A$, $(f_n(c))$ converges to some l. Define f, by $f(c) = \lim_{n\to\infty} f_n(c)$, $\forall \in A$.

Thus we showed that $f_n \xrightarrow{A} f$.

Now since (f_n) is uniformly Cauchy on A, $\forall \varepsilon > 0$ $\exists N$ s.t. $\forall m, n \geq N$, $\forall x \in A$ $|f_m(x) - f_n(x)| < \varepsilon/2$. For each $x \in A$, we take $\lim_{n \to \infty} |f_m(x) - f_n(x)| = |f_m(x) - f(x)|$ By the limit order thm. $|f_m(x) - f(x)| \leq \varepsilon/2 < \varepsilon$. Thus $\forall m \geq N$ and $\forall x \in A$, $|f_m(x) - f(x)| < \varepsilon \Rightarrow$ $f_m \overset{A}{\rightrightarrows} f$.

6.2.9. a) Let $f_n \overset{A}{\rightrightarrows} f$ and $g_n \overset{A}{\rightrightarrows} g$.

Then, given $\varepsilon > 0$ choose N_1 so that $\forall n \geq N_1$ and $\forall x \in A$, $|f_n(x) - f(x)| < \varepsilon/2$. Also choose N_2 so that $\forall n \geq N_2$ and $\forall x \in A$, $|g_n(x) - g(x)| < \varepsilon/2$. Let $N = \max\{N_1, N_2\}$. Then $\forall n \geq N$, $x \in A$,

$|f_n(x) + g_n(x) - (f(x) + g(x))| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Thus $(f_n + g_n) \overset{A}{\rightrightarrows} (f + g)$.

b) Let $f_n(x) = x$ $\forall n \in \mathbb{N}$, then $f_n \overset{\mathbb{R}}{\rightrightarrows} x = f(x)$.

Let $g_n(x) = \frac{1}{n}$ $\forall n \in \mathbb{N}$, then $g_n \overset{\mathbb{R}}{\rightrightarrows} 0 = g(x)$.

Yet $f_n(x) \cdot g_n(x) = \frac{x}{n} \underset{\text{pointwise}}{\xrightarrow{\mathbb{R}}} f(x) \cdot g(x) = 0$, but not uniformly. To see this, let $x_n = n$, let $\varepsilon = 1$ then $|f_n(x_n) \cdot g_n(x_n) - f(x_n) g(x_n)| = |\frac{n}{n} - 0| = 1 \geq \varepsilon = 1$.

Suppose $f_n \overset{A}{\rightrightarrows} f$ and $g_n \overset{A}{\rightrightarrows} g$.

c) Given $\varepsilon > 0$ (and $\varepsilon < 2M$) choose

N_1 s.t. $\forall n \geq N_1$, $\forall x \in A$, $|f_n(x) - f(x)| < \frac{\varepsilon}{2M}$.

Also choose N_2, s.t. $\forall n \geq N_2$, $\forall x \in A$,

$|g_n(x) - g(x)| < \frac{\varepsilon}{2(M+1)}$. Let $N = \max\{N_1, N_2\}$,

then $\forall n \geq N$ and $\forall x \in A$ one has

$$|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)| \leq 1 + M.$$

Moreover, $\forall n \geq N$ and $\forall x \in A$ one has

$$|f_n(x) \cdot g_n(x) - f(x) \cdot g(x)| = |f_n(x) \cdot g_n(x) - f_n(x) \cdot g(x)$$
$$+ f_n(x) \cdot g(x) - f(x) \cdot g(x)| \leq |f_n(x)| \cdot |g_n(x) - g(x)|$$
$$+ |g(x)| \cdot |f_n(x) - f(x)| \leq (M+1) \cdot \frac{\varepsilon}{2(M+1)} + M \cdot \frac{\varepsilon}{2M} = \varepsilon.$$