Homework # 30 (Solutions).

1

5.3.5 a) Let $h(x) = [f(b) - f(a)]\, g(x) - [g(b) - g(a)]\, f(x)$.

We will apply Rolle's theorem: since both $f(x)$ and $g(x)$ are continuous on $[a, b]$ and diff. on (a, b) $\Rightarrow$ $h(x)$ is also cont. on $[a,b]$ and diff. on (a,b).

$h(a) = [f(b) - f(a)]\, g(a) - [g(b) - g(a)]\, f(a)$

$= f(b)\, g(a) - g(b)\, f(a)$,

$h(b) = [f(b) - f(a)]\, g(b) - [g(b) - g(a)]\, f(b)$

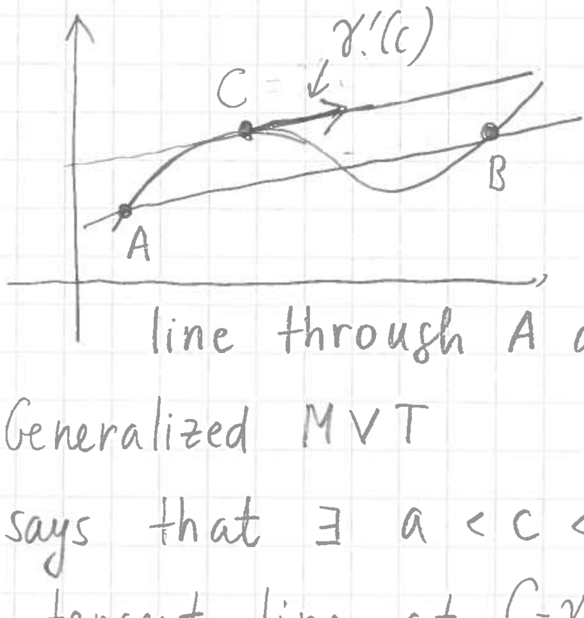
$= -f(a)\, g(b) + g(a)\, f(b)$, so

$h(a) = h(b)$. Then by Rolle's theorem

$\exists\, c \in (a, b)$ s.t. $0 = h'(c) = [f(b) - f(a)]\, g'(c) +$

$+ [g(b) - g(a)]\, f'(c)$.

b) Let $\gamma(t) = (f(t), g(t))$, $a \le t \le b$



Let $A = (f(a), g(a))$ and $B = (f(b), g(b))$

Then the slope of the line through A and b is $m = \dfrac{g(b) - g(a)}{f(b) - f(a)}$.

Generalized MVT says that $\exists\; a < c < b$ s.t. slope of the tangent line at $C = \gamma(c)$ is m. This slope is $\dfrac{g'(c)}{f'(c)}$.

2] Negation of definition 6.2.1 and 6.2.3:

- Let $f_n : A \to \mathbb{R}$, $n \in \mathbb{N}$. Sequence f_n does not converge pointwise to a function $f : A \to \mathbb{R}$ if there exists a $c \in A$ and an $\varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$ $\exists n \geq N$ s.t. $|f_n(c) - f(c)| \geq \varepsilon$.

- Sequence f_n does not converge uniformly to a function $f : A \to \mathbb{R}$ if $\exists \varepsilon > 0$ such that $\forall N \in \mathbb{N}$ $\exists n \geq N$ and $\exists x \in A$ so that $|f_n(x) - f(x)| \geq \varepsilon$

3] Observe that $\lim\limits_{n\to\infty} x^n = \begin{cases} 0, & \text{if } -1 < x < 1 \\ 1, & \text{if } x = 1 \\ +\infty, & \text{if } x > 1. \end{cases}$

a) $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+x^n} = \begin{cases} x, & \text{if } 0 \leq x < 1 \\ 1/2, & \text{if } x = 1 \\ 0, & x > 1 \end{cases}$$

b) $$\lim_{n\to\infty} g_n(x) = \begin{cases} \lim\limits_{n\to\infty} \dfrac{\cancel{n}\left(\frac{x^n}{n} - 1\right)}{\cancel{n}\left(\frac{x^n}{n} + 1\right)} = -1, & -1 < x \leq 1 \\ \lim\limits_{n\to\infty} \dfrac{\cancel{x^n}\left(1 - \frac{n}{x^n}\right)}{\cancel{x^n}\left(1 + \frac{n}{x^n}\right)} = 1, & \text{if } x > 1. \end{cases}$$

c) $$\lim_{n\to\infty} h_n(x) = \begin{cases} 0, & x = 0 \\ \lim\limits_{n\to\infty} \dfrac{\cancel{n}(x^2)}{\cancel{n}\left(\frac{1}{n} + x^3\right)} = \dfrac{x^2}{x^3} = \dfrac{1}{x}, & x > 0, \end{cases}$$