Homework #2 (Solutions)

#1 a) Given $x \in \mathbb{Q}$,

there exists $(-x) \in \mathbb{Q}$ s.t $(-x) + x = 0$ (A5)

$$[(-x) + x] \cdot y = 0 \cdot y \text{ (Property of equality)}$$

$$(-x) \cdot y + x \cdot y = 0 \cdot y \text{ (D1)}$$

We showed in class that $0 \cdot y = 0$, so

$$(-x) \cdot y + xy = 0$$

There exists $-(x \cdot y) \in \mathbb{Q}$ s.t. $-(x \cdot y) + xy = 0$ (A5)

$$((-x) \cdot y + x \cdot y) + (-x \cdot y) = 0 + (-x \cdot y) \text{ (Property of equality)}$$

$$(-x) \cdot y + (x \cdot y + (-x \cdot y)) = -x \cdot y \text{ (A4 and A2)}$$

$$(-x) \cdot y + 0 = -x \cdot y \text{ (A5)}$$

$$(-x) \cdot y = -x \cdot y \text{ (A4).}$$

b) Given $z \in \mathbb{Q}$, $z \neq 0$, there exists a multiplicative inverse z^{-1} s.t. $z \cdot z^{-1} = 1$ (M5)

$$(x \cdot z) \cdot z^{-1} = (y \cdot z) z^{-1} \text{ (Property of equality)}$$

$$x \cdot (z \cdot z^{-1}) = y \cdot (z \cdot z^{-1}) \text{ (M3)}$$

$$x \cdot 1 = y \cdot 1 \quad \text{(M5)}$$

$$x = y \quad \text{(M4).}$$

#1.2.3 (a) This statement is false. Counterexample:

let $A_n = [n, +\infty)$, $n = 1, 2, 3, \ldots$. Then clearly $A_1 \supset A_2 \supset \ldots$ and $\bigcap_{n=1}^{\infty} A_n = \emptyset$

(b) This statement is true. If $A_1 \supseteq A_2 \supseteq \ldots$ are all finite and nonempty, then the is a $k \in \mathbb{N}$, so that $A_k = A_{k+2} = \ldots$ (that is for m and $n \geq k$, $A_m = A_n$, i.e. the sets no longer change)
Then $\bigcap_{n=1}^{\infty} A_n = A_k \leftarrow$ finite and nonempty.

We can prove that k exists by contradiction. Suppose there is no such k, then there is a set A_{n_1} so that $A_{n_1} \neq A_1$, let $a_1 \in A_1$, $a_1 \notin A_{n_1}$. Similarly there is a set $A_{n_2} \subset A_{n_1}$ such that $A_{n_2} \neq A_{n_1}$, let $a_2 \in A_{n_1}$, $a_2 \notin A_{n_2}$, $a_2 \neq a_1$. We can continue this process obtaining an infinite sequence $a_1, a_2, \ldots, a_n, \ldots$ such that $a_i \neq a_j$ and $a_n \in A_1$ for all n. This is a contradiction since a finite set A_1 can not contain an infinite sequence of distinct elements.

(c) This statement is false. Counterexample:

Let $A = [0,1]$, $B = [1,2]$, $C = [1,2]$.
Then $A \cap (B \cup C) = [0,1] \cap [1,2] = \{1\}$ and

$(A \cap B) \cup C = \{1\} \cup [1,2] = [1,2]$. (In fact, $A \cap (B \cup C)$ is a subset of $(A \cap B) \cup C$)

(d) True. $x \in A \cap (B \cap C) \Leftrightarrow x \in A$ or $(x \in B$ or $C) \Leftrightarrow x \in A$ or $x \in B$ or $x \in C$ $\Leftrightarrow (x \in A$ or $x \in B)$ or $x \in C \Leftrightarrow x \in (A \cap B) \cap C$.

(e) True.
$x \in A \cap (B \cup C) \Leftrightarrow x \in A$ and $(x \in B$ or $x \in C) \Leftrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ or $x \in C) \Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

#1.2.4 Observe that $\forall n \in \mathbb{N}$, (*)

$n = 2^k \cdot m$ where m is an odd number.

Let $A_1 = \{ m \mid m \text{ is odd} \}$

$A_2 = \{ 2 \cdot m \mid m \text{ is odd} \}$

. . . . . .

$A_n = \{ 2^{n-1} \cdot m \mid m \text{ is odd} \}$.

. . . .

Then sets A_n are infinite,

$A_n \cap A_k = \emptyset$ if $n \neq k$, since

$x \in A_n \cap A_k \Rightarrow x = 2^n m_1 = 2^k m_2 \Rightarrow k = n$ and $m_1 = m_2$.

Moreover, $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$ by the observation (*).