Homework # 29 (Solutions)

#1 a) $[a,b]$ is a compact set. If $f'(x)$ is continuous on $[a,b]$, then $f'(x)$ is bounded on $[a,b]$ by thm 4.4.1 ($f'([a,b])$ is compact, and thus bounded), i.e. $\exists M>0$ s.t. $|f'(x)| \leq M$ $\forall x \in [a,b]$.

b) By Mean Value Thm (MVT)

$\forall x, y \in [a,b]$, $\exists c$ between x and y s.t.

$f(x) - f(y) = f'(c)(x-y)$.

Then $|f(x) - f(y)| = |f'(c)| \cdot |x-y| \leq M \cdot |x-y|$.

c) By MVT, $\forall x, y \in \mathbb{R}$, $\exists c$ between x and y s.t. $f(x) - f(y) = f'(c)(x-y)$

Then since $f'(x)$ is bounded on $\mathbb{R}$, $\exists$ K s.t. $|f'(c)| \leq K$ $\forall c \in \mathbb{R}$.

Then $|f(x) - f(y)| \leq K|x-y|$.

Now, given $\varepsilon > 0$, choose $\delta = \varepsilon/K$. Then

$|x-y| < \delta \Rightarrow |f(x) - f(y)| \leq K \cdot |x-y| < \varepsilon/K \cdot K = \varepsilon$.

Thus, $f(x)$ is uniformly continuous on $\mathbb{R}$.

d) $\Rightarrow$ Let $x > y$, then $f'(c) \geq 0$ $\forall c \in \mathbb{R}$

$\Rightarrow$ by MVT that $f(x) - f(y) = f'(c)(x-y) \geq 0$

$\Rightarrow f(x) \geq f(y)$ $\forall x > y$, so $f(x)$ is increasing.

$\Leftarrow$ Suppose $f(x)$ is increasing, then $\forall\, x < y$, $f(x) \le f(y)$. Then $\frac{f(y)-f(x)}{y-x} \ge 0$, so

So $f'(x) = \lim_{y \to x} \frac{f(y)-f(x)}{y-x} \ge 0$ (by the algebraic limit thm.

5.3.2. Proof by contradiction.

Suppose $f'(x) \ne 0$ on A and yet f is not one-to-one. Then $\exists\, a, b \in A$ s.t. $a \ne b$ and $f(a) = f(b)$. By MVT, $\exists\, c \in A$ s.t.

$0 = f(b) - f(a) = f'(c)(b-a) \Rightarrow f'(c) = 0$, contradiction.

Converse is not true: $f(x) = x^3$ is one-to-one on $[-1, 1]$, yet $f'(0) = 0$.

5.3.3 a) Since h is differentiable on $[0,3]$, h is continuous on $[0,3]$.

Consider the function $g(x) = h(x) - x$ which is also continuous on $[0,3]$.

$g(0) = h(0) - 0 = 1$ and $g(3) = h(3) - 3 = -1$

Intermediate value thm applied on $[0,3]$ guarantees that $\exists\, d \in (0,3)$ such that $g(d) = 0$, i.e. $h(d) = d$.

b) Apply the Mean Value theorem on $[0,3]$.

$\underbrace{h(3) - h(0)}_{1} = h'(c)(3-0)$, $h'(c) = \frac{1}{3}$, $c \in (0,3)$

c) Apply the Rolle's thm on $[1,3]$, to conclude that $\exists\, r \in (1,3)$ with $h'(r) = 0$.

Now apply Darboux thm to an interval with endpoints at c and r

Since $0 = h'(r) < \frac{1}{4} < \frac{1}{3} = h'(c)$, $\exists\, x$ on this interval with $h'(x) = \frac{1}{4}$