Homework # 28 (Solutions)

1 a)

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{\frac{1}{x^2}-\frac{1}{c^2}}{x-c} =$$

$$= \lim_{x\to c} \frac{c^2-x^2}{x^2c^2(x-c)} = \lim_{x\to c} \frac{-(x+c)}{x^2c^2} = -\lim_{x\to c} \frac{2c}{c^4}$$

$$= -\frac{2}{c^3}.$$

b) Observe that $g(x) = \begin{cases} x^2, & x \geq 0, \\ -x^2, & x < 0. \end{cases}$

If $c > 0 \Rightarrow g'(c) = \lim_{x\to c} \frac{x^2-c^2}{x-c} = 2c$,

· if $c < 0 \Rightarrow g'(c) = -2c$

· if $c = 0 \Rightarrow g'(0) = \lim_{x\to 0} \frac{x|x|}{x} = 0.$

5.2.2. a) $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$, $g(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$

f and g are not continuous at $x=0$, so also not differentiable

b) Let $f(x)$ be as in a) and $g(x) = 0$,
then $f \cdot g = 0$.

c) Not possible : $f = (f+g) - g$, if $f+g$ and g are both differentiable, then f is also differentiable by thm. 5.2.4 (i) and (ii).

d) Let $f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$.

Observe that $f(x)$ is differentiable at $x=0$ since

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \left(\begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases} \right) = 0.$$

f is <u>not</u> differentiable at any $c \neq 0$ since f is not even continuous at $c \neq 0$.

$\boxed{\#5.2.5}$

a) $\lim\limits_{x\to 0} f_a(x) = \begin{cases} 0 & \text{if } a>0 \\ \text{DNE} & \text{if } a \le 0 \end{cases}$ So $f_a(x)$ is continuous at $x=0$ when $a>0$

To see what happens if $a \le 0$, consider two sequences

$x_n = \frac{1}{n}$ and $y_n = -\frac{1}{n}$, then

$$\lim f_a(x_n) = \lim \frac{1}{n^\alpha} = \begin{cases} +\infty, & \alpha<0 \\ 1, & \alpha=0 \end{cases}$$

$$\lim f_a(y_n) = 0$$

for $\alpha \le 0$

So since $\lim f_a(x_n) \neq \lim f_a(y_n)$, $\lim\limits_{x\to 0} f_a(x)$ DNE.

b) For $a>0$

$$\lim_{x\to 0} \frac{f_a(x)-f_a(0)}{x} = \lim_{x\to 0} \begin{cases} x^{a-1}, & x>0 \\ 0, & x<0 \end{cases} = \begin{cases} 0, & \text{if } a-1>0 \\ \text{DNE}, & \text{if } a-1 \le 0. \end{cases}$$

So, $f_a'(0)$ exists only when $a>1$, and for $a>1$ we have

$$f_a'(x) = \begin{cases} a x^{a-1}, & x \geq 0 \\ 0, & x<0. \end{cases}$$

c) For $a > 1$

$$\lim_{x\to 0} \frac{f_a'(x) - f_a'(0)}{x} = \lim_{x\to 0} \begin{cases} a x^{a-2}, & x > 0 \\ 0, & x < 0 \end{cases}$$

$$= \begin{cases} 0, & \text{if } a-2>0 \\ \text{DNE}, & \text{if } a-2 \le 0 \end{cases}$$

So, $f_a''(0)$ exists only when $a > 2$.