Hwk # 27 (Solutions)

1 a) Not uniformly continuous: Apply thm 4.4.5

Let $x_n = \frac{1}{2\pi n}$, $y_n = \frac{1}{\pi + 2\pi n}$, let $\varepsilon = 1$

Then $\lim |x_n - y| = 0$, but $|f(x_n) - f(y_n)| =$

$= |\cos(2\pi n) - \cos(\pi + 2\pi n)| = 2 > 1$.

(b) $g(x)$ is uniformly continuous on $[0, 1]$:

Observe that $\lim_{x \to 0} g(x) = 0$, so g is continuous at $x = 0$. For $\forall x \neq 0$, $g(x)$ is continuous as a product of two continuous functions. Thus $g(x)$ is cont. on $[0, 1]$. Since $[0, 1]$ is compact $\Rightarrow g(x)$ is uniformly continuous on $[0, 1]$ by thm. 4.4.7

c) $h(x) = \sqrt{x}$ is uniformly continuous on $[0, +\infty)$.

Since $h(x)$ is continuous on $[0, +\infty)$,

$h(x)$ is uniformly continuous on $[0, 1]$ by thm 4.4.7.

We will show that $h(x)$ is uniformly continuous on $[1, +\infty)$: given $\varepsilon > 0$ choose $\delta = \varepsilon$, then $\forall x, y \in [1, +\infty)$, $|x - y| < \delta \Rightarrow$

$$|\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{\delta}{2} < \varepsilon.$$

By the exercise 4.4.5, $\sqrt{x}$ is uniformly continuous on $[0, 1] \cup [1, +\infty)$.

#4.4.5. Suppose $\varepsilon > 0$ is given. Since g is uniformly continuous on $(a,b]$ $\exists\ \delta_1 > 0$ such that $\forall\ x,y \in (a,c]$ with $|x-y| < \delta_1$ we have $|g(x)-g(y)| < \frac{\varepsilon}{2}$.
Similarly, since g is uniformly continuous on $[b,c)$ $\exists\ \delta_2 > 0$ s.t. $\forall\ x,y \in [b,c)$ with $|x-y| < \delta_2$ we have $|g(x)-g(y)| < \frac{\varepsilon}{2}$. Let $\delta = \min\{\delta_1, \delta_2\}$, then $\forall\ x,y \in (a,c]$ with $|x-y| < \delta$ we have

- if $x,y \in (a,b] \Rightarrow |g(x)-g(y)| < \frac{\varepsilon}{2} < \varepsilon$
- if $x,y \in [b,c) \Rightarrow |g(x)-g(y)| < \frac{\varepsilon}{2} < \varepsilon$
- if x is in $(a,b]$ and $y \in [b,c)$ (or $y \in (a,b]$, and $x \in [b,c)$) then $|x-y| < \delta \Rightarrow |x-b| < \delta$ and $|y-b| < \delta$, so $|g(x)-g(y)| = |g(x)-g(b)+g(b)-g(y)| \leq$
$\leq |g(x)-g(b)| + |g(b)-g(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thus, in all cases $|x-y| < \delta \Rightarrow |g(x)-g(y)| < \varepsilon$.

4.4.8 a) Suppose b is such that $f(x)$ is uniformly continuous on $[b,+\infty)$. $f(x)$ is also uniformly continuous on $[0,b]$, since $f(x)$ is continuous on compact $[0,b]$. By the argument we already made in #4.4.7 $f(x)$ is uniformly continuous on $[0,+\infty)$.

b) $f(x) = \sqrt{x}$ is continuous on $[0,\infty)$, it is uniformly continuous on $[0,1]$. We will show that

#4.4.6 (a) Yes, this is possible. Take $(x_n) = (\frac{1}{n})_{n=2}^{\infty}$

(x_n) is Cauchy, since $\lim x_n = 0$. Yet if $f(x) = \frac{1}{x}$,

$(f(x_n)) = (n) \leftarrow$ not Cauchy, since it is not bounded.

(b) This is impossible: if (x_n) is Cauchy, (x_n) converges to some l. Since $(x_n) \subset [0, 1]$ and $[0,1]$ is closed, $l \in [0,1]$. Since f is continuous on $[0,1]$, f is continuous at l and $\lim f(x_n) = f(l)$, so sequence $(f(x_n))$ also must be Cauchy, since it converges.

(c) This is impossible. If (x_n) is Cauchy, it converges to some $l \in [0, +\infty)$ as in (b). Then $f(x_n)$ converges to $f(l)$ by continuity of f, so $(f(x_n))$ is Cauchy.

#4.5.7

Consider $g(x) = f(x) - x$. Observe that $g(x)$ is continuous on $[a,b]$ and since (range of $f(x)) \subseteq [0, 1]$

$\Rightarrow g(0) \geq 0 - 0 = 0$ and $g(1) \leq 1 - 1 = 0$

Thus $g(0) \geq 0$ and $g(1) \leq 0 \Rightarrow \exists\, c \in [0, 1]$ s.t. $g(c) = 0 \Leftrightarrow f(c) = c$, so c will be a fixed point of f.