Homework 26 solutions

Sunday, November 11, 2018 10:32 AM

1. a), c) Given $\varepsilon > 0$, choose $\delta = \varepsilon/2$, then $|x-y| < \delta$, $x, y \geq 1$

$$\Rightarrow |f(x) - f(y)| = \left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \left| \frac{(x+y)(x-y)}{x^2 y^2} \right|$$

$$= \left| \frac{x+y}{x^2 y^2} \right| |x-y| = \left| \frac{1}{x^2 y} + \frac{1}{x y^2} \right| |x-y| < 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

$\Rightarrow$ $f(x)$ is uniformly continuous on both $[1,3]$ and $[1,\infty)$.

b) Not uniformly continuous: Let $(x_n) = (\frac{1}{n})$ and $(y_n) = (\frac{1}{2n})$. Then $\lim(x_n - y_n) = 0$, yet $|f(x_n) - f(y_n)| = n \geq 1$.

2. Given $\varepsilon > 0$, choose $\delta = \varepsilon/M$. Then $|x-y| < \delta \Rightarrow$ $|f(x) - f(y)| \leq M|x-y| < M \cdot \delta = \varepsilon$.

\# 4.4.11 $\Rightarrow$ Suppose $g: \mathbb{R} \to \mathbb{R}$ is continuous. Let O be an open set, consider $g^{-1}(O)$ and assume $g^{-1}(O) \neq \phi$. Let $c \in g^{-1}(O)$, then $g(c) \in O$. Since O is open, $\exists\, \varepsilon > 0$ such that $V_\varepsilon(g(c)) \subseteq O$. Since g is continuous, $\exists\, V_\delta(c)$ with $f(V_\delta(c)) \subseteq V_\varepsilon(g(c))$, i.e. $\forall\, c \in g^{-1}(O)$ $\exists\, V_\delta(c) \subseteq g^{-1}(O)$, so $g^{-1}(O)$ is open.

$\Leftarrow$ Suppose $g: \mathbb{R} \to \mathbb{R}$ has a property that $g^{-1}(O)$ is open for each open O.
Let $c \in \mathbb{R}$ be arbitrary and given any $\varepsilon > 0$, the set $V_\varepsilon(f(c))$ is open and thus the set $f^{-1}(V_\varepsilon(f(c))$ is also open and contains c.

Thus $\exists\, \delta$ s.t. $V_\delta(c) \subseteq f^{-1}(V_\varepsilon(f(c))$, i.e. $f(V_\delta(c)) \subseteq V_\varepsilon(f(c))$, and f is continuous at c by def. 4.2.1B.