Homework 25 solutions

Sunday, November 11, 2018 9:29 AM

1. In order to show that $f \cdot g$ is continuous at $x=c$, choose arbitrary $\varepsilon > 0$. Then,

since $f(x)$ is continuous at $x=c$, given $\varepsilon_1 = 1$

choose $\delta_1 > 0$ s.t. $|x-c| < \delta_1$, $x \in A \Rightarrow |f(x) - f(c)| < 1$.

$\Rightarrow |f(x)| < |f(c)| + 1$.

Also, given $\varepsilon_2 = \frac{\varepsilon}{2(|g(c)|+1)}$, choose $\delta_2 > 0$ s.t

$|x-c| < \delta_2$, $x \in A \Rightarrow |f(x) - f(c)| < \varepsilon_2$.

Since g is continuous at $x=c$, given $\varepsilon_3 = \frac{\varepsilon}{2(f(c)+1)}$,

choose $\delta_3 > 0$ s.t. $|x-c| < \delta_3$, $x \in A \Rightarrow |g(x) - g(c)| < \varepsilon_3$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then $\forall x \in A$,

$|x-c| < \delta \Rightarrow |f(x)g(x) - f(c)g(c)| = |f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)| \leq$

$|f(x)| \cdot |g(x) - g(c)| + |g(c)| \cdot |f(x) - f(c)| <$

$(|f(c)|+1) \cdot \varepsilon_2 + (|g(c)|+1) \cdot \varepsilon_3 = \varepsilon/2 + \varepsilon/2 = \varepsilon$.

2. Function f is not uniformly continuous on the set A if there exists an $\varepsilon > 0$, such that for every $\delta > 0$, there exist $x, y \in A$ s.t. $|x-y| < \delta$, yet $|f(x) - f(y)| \geq \varepsilon$.

3. 4.3.3 a) Given $\varepsilon > 0$, since g is continuous at $y = f(c)$, choose $\delta_1 > 0$, s.t. $|y - f(c)| < \delta_1 \Rightarrow |g(y) - g(f(c))| < \varepsilon$.

Since f is continuous at $x=c$, given $\varepsilon_1 = \delta_1$, choose $\delta > 0$, s.t. $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \delta_1$.

Then $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \delta_1 \Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$.

b) Let $(x_n) \subset A$ be an arbitrary sequence s.t. $\lim x_n = c \Rightarrow$ by continuity of f at $x=c$

$\lim f(x_n) = f(c) \Rightarrow$ by continuity of g at $f(c)$

$\lim g(f(x_n)) = g(f(c))$. Then $g \circ f$ is continuous at $x=c$ by the sequential criterion of continuity.

4.3.6 (a) Let $f(x) = \begin{cases} 1, & x \neq 0 \\ -1, & x = 0 \end{cases}$, $g(x) = \begin{cases} -1, & x \neq 0 \\ 1, & x = 0 \end{cases}$.

Then $f(x) + g(x) = 0$, $f(x) \cdot g(x) = -1$.

(b) If $f(x)$ and $f(x) + g(x)$ are both continuous at $x=0$, then by the Algebraic Continuity thm $g(x) = f(x) + g(x) - f(x)$ is also cont. at 0.

(c) Let $f(x) = 0$, then let $g(x)$ be as in (a), then $f(x) \cdot g(x) = 0$.

d) Let $f(x) = \begin{cases} 2, & x \neq 0 \\ 1/2, & x = 0 \end{cases}$, then $f(x) + \frac{1}{f(x)} = \begin{cases} 2 + 1/2, & x \neq 0 \\ 1/2 + 2, & x = 0 \end{cases}$,

so $f(x) = 2.5$ is continuous.

e) Not possible. If $g(x) = [f(x)]^3$ is continuous, then $f(x) = h(g(x))$ is continuous, where $h(x) = x^{1/3}$.