Homework # 24.

 # 2 a) Since $\lim_{x \to c} f(x) = L$ and $L \neq 0$,

choose $\varepsilon_1 = \frac{|L|}{2}$ and find δ_1 s.t $0<|x-c|<\delta_1$,

$x \in A \implies |f(x) - L| < \varepsilon$.

Then $|L| - |f(x)| \leq |f(x) - L| < \frac{|L|}{2} \implies$

$|f(x)| > |L| - \frac{|L|}{2} = \frac{|L|}{2}$.

b) Given $\varepsilon > 0$, let $\varepsilon_2 = \frac{\varepsilon \cdot 2}{|L|^2}$, choose $\delta_2 > 0$

s.t. $\forall x \in A$, $0 < |x-c| < \delta_2 \implies |f(x) - L| < \varepsilon_2$.

Then let $\delta = \min\{\delta_1, \delta_2\}$, $\forall x \in A$, $0<|x-c|<\delta$

$$\implies \left|\frac{1}{f(x)} - \frac{1}{L}\right| = \frac{|f(x) - L|}{|f(x)| \cdot |L|} < \frac{\varepsilon_2}{\frac{|L|}{2} \cdot |L|} = \varepsilon.$$

 # 4.2.9. a) Let $f: A \to \mathbb{R}$ and let c be a limit point of A. We say that $\lim_{x \to c} f(x) = \infty$ if

$\forall M > 0 \ \exists \delta > 0$ such that $\forall x$, $0 < |x - c| < \delta$, $x \in A$, we have $f(x) > M$.

Given $M > 0$, choose $\delta = \frac{1}{\sqrt{M}}$. When $0<|x|<\delta$,

we have $\frac{1}{x^2} > \frac{1}{\delta^2} = M$.

b) Let $f: A \to \mathbb{R}$, A extends to $+\infty$, We say $\lim_{x \to \infty} f(x) = L$

$\forall \varepsilon > 0 \ \exists R > 0$ s.t. $\forall x > R$, $x \in A$, we have $|f(x) - L| < \varepsilon$.

Given $\varepsilon > 0$ choose $R = \frac{1}{\varepsilon}$, then for $x > R$, $x \neq 0$,
$|f(x) - 0| = \frac{1}{x} < \frac{1}{R} = \varepsilon$.

4.3.1 (a) Given $\varepsilon > 0$, choose $\delta = \varepsilon^3$, then for all x s.t. $|x - c| < \delta$ we have
$|g(x) - g(c)| = |\sqrt[3]{x}| < \delta^{1/3} = \varepsilon$.

b) Preliminary work: first assume that $c > 0$, then if $|x - c| < \frac{|c|}{2}$, then $x > c - \frac{|c|}{2} > 0$.

$$|\sqrt[3]{x} - \sqrt[3]{c}| = \frac{|(x^{1/3} - c^{1/3})(x^{2/3} + x^{1/3}c^{1/3} + c^{2/3})|}{|x^{2/3} + x^{1/3}c^{1/3} + c^{2/3}|} \leq \frac{|x - c|}{c^{2/3}}$$

since all terms in the denominator are positive.

Now given $\varepsilon > 0$ choose $\delta = \min\left\{\frac{|c|}{2}, \varepsilon \cdot c^{2/3}\right\}$, then for all x such that $|x - c| < \delta$ we have

$$|\sqrt[3]{x} - \sqrt[3]{c}| \leq \frac{|x - c|}{c^{2/3}} < \frac{\delta}{c^{2/3}} \leq \frac{\varepsilon \cdot c^{2/3}}{c^{2/3}} = \varepsilon.$$

We showed that $g(x) = \sqrt[3]{x}$ is continuous $\forall c \in [0, +\infty)$. To show that $g(x)$ is also continuous for all $c < 0$, observe that if $c < 0$, then

$$\lim_{x \to c} g(x) \overset{y=-x}{=} \lim_{y \to -c} g(-y) = -\lim_{y \to \underbrace{-c}_{\text{positive}}} g(y) = -g(-c) = g(c).$$