Homework # 23 (Solutions)

1 a) Let $O_n = (-n, n)$. Then $\{O_n \mid n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$ since $\bigcup_{n=1}^{\infty} O_n = \mathbb{R}$.

Suppose $O_{n_1}, \ldots, O_{n_k}$ with $n_1 < \ldots < n_k$ is any finite collection, then

$\bigcup_{j=1}^{k} O_{n_j} = O_{n_k} = (-n_k, n_k) \neq \mathbb{R}$. So, no finite collection covers $\mathbb{R}$.

b) Let $O_n = V_{1/2}(n^2)$. Then $\{O_n \mid n \in \mathbb{N}\}$ is an open cover. If even one set O_k is removed from this open cover, then point k^2 is no longer covered. Thus this open cover has no finite subcover.

c) Let $O_n = (0, \frac{2}{n})$, $n \in \mathbb{N}$.

Since $\bigcup_{n=1}^{\infty} O_n = (0, 2) \supset \{\frac{1}{n}, n \in \mathbb{N}\}$, $\{O_n, n \in \mathbb{N}\}$ is an open cover.

Suppose $O_{n_1}, \ldots, O_{n_k}$ with $n_1 < \ldots < n_k$ is any finite collection, then $\bigcup_{j=1}^{k} O_{n_j} = O_{n_k} = (0, 2/n_k) \not\ni \frac{1}{n_k}$.

So, no finite collection of O_n's can cover $\{\frac{1}{n}; n \in \mathbb{N}\}$.

d) Let $O_n = (-1, 1/n)$, then $\bigcup_{n=1}^{\infty} O_n = (-1, 1) \supset [0, 1)$.

If $O_{n_1}, \ldots, O_{n_k}$ is any finite collection, then

$$\bigcup_{j=1}^{k} O_{n_j} = \left(-1, \frac{1}{n_k}\right) \leftarrow \text{does not cover } \frac{1}{n_k+1},$$

so no finite subcover exists.

Homework # 23 (Solutions)

#3.3.8 (a) Suppose both $A_1 \cap K$ and $B_1 \cap K$ have finite subcover consisting of sets from $\{O_\lambda : \lambda \in \Lambda\}$. Then by combining both finite subcovers, we obtain a finite subcover of $(A_1 \cap K) \cup (B_1 \cap K) = K$. This is a contradiction.

(b) Since K is compact, it is bounded. If $\forall x \in K$, $|x| \le M$, we can take $I_0 = [-M, M]$. Now we will apply induction. Suppose $I_0 \supseteq I_1 \supseteq \dots \supseteq I_k$ are such that $I_j \cap K$ can not be finitely covered and $|I_j| = \frac{|I_0|}{2^j}$ $\forall j = 1, \dots, k$. Divide I_k into two halfs, call them A_{k+1} and B_{k+1} and apply (a). If $A_{k+1} \cap K$ cannot be finitely covered, then $I_{k+1} = A_{k+1}$, otherwise $B_{k+1} \cap K$ cannot be finitely covered and then $I_{k+1} = B_{k+1}$. $|I_{k+1}| = \frac{|I_k|}{2} = \frac{|I_0|}{2^{k+1}}$.

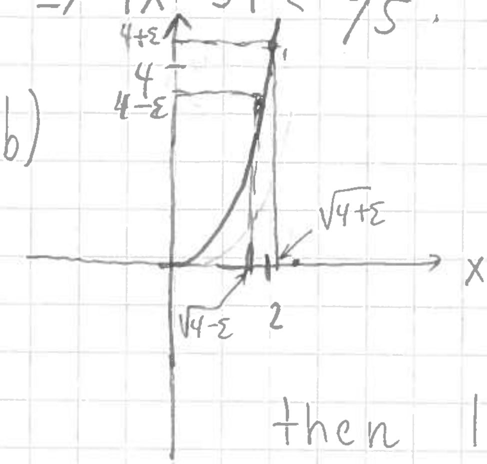
(c) Sets $I_n \cap K$ are nested nonempty compact sets. Thus $\bigcap_{n=1}^{\infty} (I_n \cap K) \neq \emptyset$.

(d) Let $x \in \bigcap_{n=1}^{\infty} (I_n \cap K)$. Since $x \in K$, x is covered by some open set O_{λ_0}. Since O_{λ_0} is open, $\exists V_\varepsilon(x) \subseteq O_{\lambda_0}$. Then if $|I_{n_0}| < \varepsilon$, $I_{n_0} \subset V_\varepsilon(x)$ since $x \in I_{n_0}$ and $x \in V_\varepsilon(x)$. Then the set $I_{n_0} \cap K$ is covered with O_{λ_0} which is a contradiction with assumption that $I_{n_0} \cap K$ cannot be finitely covered ($|I_{n_0}| = \frac{|I_0|}{2^n} < \varepsilon$ will hold for $n > \frac{\ln \frac{|I_0|}{\varepsilon}}{\ln 2}$).

#3.) a) $|f(x) - L| = |5x + 7 - 22| = 5|x-3| < \varepsilon$

$\Rightarrow |x-3| < \varepsilon/5$. Thus $\delta_{max}(\varepsilon) = \varepsilon/5$.

b)



From the picture, if $|x-2| < \delta_{min}$, where $\delta_{min} = \min\{\sqrt{4+\varepsilon} - 2, 2 - \sqrt{4-\varepsilon}\}$, then $|x^2 - 4| < \varepsilon$.

Observe that $\sqrt{4+\varepsilon} - 2 < 2 - \sqrt{4-\varepsilon}$, since

$\sqrt{4+\varepsilon} - 2 < 2 - \sqrt{4-\varepsilon} \Leftrightarrow \sqrt{4+\varepsilon} + \sqrt{4-\varepsilon} < 4 \Leftrightarrow$

$8 + 2\sqrt{4-\varepsilon^2} < 16 \Leftrightarrow \sqrt{4-\varepsilon^2} < 4$, so

$$\boxed{\delta_{min} = \sqrt{4+\varepsilon} - 2}$$