Homework # 21 (Solutions)

1. We will check definition of compact. Assume $K \neq \emptyset$. Let (x_n) be any sequence in K. Since K is bounded, (x_n) is also bounded. By Bolzano-Weierstrass, (x_n) has a convergent subsequence (x_{n_k}). (x_{n_k}) is Cauchy and since K is closed, converges to x in K by thm. 3.2.8.

2. We will check the definition of closed set. Suppose x is any limit point of K, then $\exists\, (x_n) \subseteq K$ s.t. $x_n \neq x$ $\forall n$ and $\lim x_n = x$. Since K is compact $\exists$ subsequence (x_{n_k}) of (x_n) that converges to an element of K. But since $\lim\limits_{k\to\infty} x_{n_k} = x \Rightarrow x \in K$, so K is closed.

3.3.1 Suppose $K \neq \emptyset$, let $s = \sup K$ (it exists since K is bdd) $\forall n$ $\exists$ $s_n \in K$ s.t $s - \frac{1}{n} < s_n \leq s$. Then $\lim s_n = s$ and since K is compact, (s_n) has a subsequence that converges to an element of K. Since this any subsequence of (s_n) converges to $s \Rightarrow s \in K$.

The proof that $\inf K \in K$ is almost identical to the proof above.

3.3.2 a) $\mathbb{N}$ is not compact: sequence $(a_n) = (n)$ has no convergent subsequence.

b) $\mathbb{Q} \cap [0,1]$ is not compact. Let $b_n \in \mathbb{Q}$ s.t. $\frac{\sqrt{2}}{2} - \frac{1}{10n} < b_n < \frac{\sqrt{2}}{2}$, then $\lim b_n = \frac{\sqrt{2}}{2} \notin \mathbb{Q} \cap [0,1]$. Thus, (b_n) does not have subsequences convergent to elts of $\mathbb{Q} \cap [0,1]$

c) The Cantor set is compact, since it is Closed and bounded.

d) Set $\{S_n\}_{n=1}^{\infty}$, where $S_n = 1 + \frac{1}{2^2} + \ldots + \frac{1}{n^2}$ is not compact. Let $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$ be the sum of this convergent series.

$\lim S_n = L$. Any subsequence of (S_n) also converges to L, yet $L \notin \{S_n\}_{n=1}^{\infty}$ since $L < S_n \ \forall n$.

e) This set has a unique limit pt 1, which is an element of the set, thus set is closed. The set is also bounded
$\Rightarrow$ it is compact.