Homework # 20 (Solutions)

3.2.5 $\Longrightarrow$ Suppose that F is closed, if $(a_n) \subseteq F$ is Cauchy and $\lim a_n = x$ with $x \notin F$.
It follows from topological def. of limit that $\forall \varepsilon > 0 \;\; V_\varepsilon(x) \cap F = V_\varepsilon(x) \cap (F - \{x\}) \neq \emptyset$, so x must be a limit point of F. Since F is closed, $x \in F$, which is a contradiction with $x \notin F$.
Thus, if $(a_n) \subseteq F$ is Cauchy and $\lim a_n = x \Rightarrow x \in F$.

$\Longleftarrow$ If x is any limit point of F, then $\exists\, (a_n) \subset F$, $a_n \neq x \;\; \forall n$ and $\lim a_n = x \Rightarrow x \in F$ by the assumption. Thus F is closed.

3.2.7 (a) Let m be an arbitrary limit point of L. Then there exists sequence $(\ell_n) \subset L$, $\ell_n \neq m$, $\lim \ell_n = m$. Since $\forall n$, ℓ_n is a limit point of A $\exists\, a_n \in A$ s.t., $a_n \in \left(V_{1/n}(\ell_n) - \{\ell_n\}\right)$
We can also choose $a_n \neq m$ (since $\ell_n \neq m$).
Then $\lim |a_n - m| \leq \lim(|a_n - \ell_n| + |\ell_n - m|) = 0 + 0$
$\Rightarrow \lim a_n = m$ and m is also a limit point of A. Thus $m \in L$, since L is the set of limit points of A! Thus, L is closed.

(b) Let now x be a limit point of $A \cup L$, then there is a sequence (x_n), $\lim x_n = x$, $(x_n) \subset A \cup L$, $x_n \neq x$. There is a subsequence (x_{n_k}) of (x_n) s.t. (x_{n_k}) is completely in A or (x_{n_k}) is completely in L.
If $(x_{n_k}) \subseteq A \Rightarrow x$ is a limit point of $A \Rightarrow x \in L$
If $(x_{n_k}) \subseteq L \Rightarrow x$ is a limit pt. of $L \Rightarrow x \in L$ by a)
$\Rightarrow x \in L \Rightarrow x \in A \cup L$.

3.2.9

(a) • $\left(x \in \left(\bigcup_{\lambda\in\Lambda} E_\lambda\right)^c\right) \Leftrightarrow \left(x \notin \bigcup_{\lambda\in\Lambda} E_\lambda\right) \Leftrightarrow (x \notin E_\lambda \text{ for all } \lambda\in\Lambda)$

$\Leftrightarrow (x \in (E_\lambda)^c \text{ for all } \lambda\in\Lambda) \Leftrightarrow \left(x \in \bigcap_{\lambda\in\Lambda} E_\lambda^c\right)$

• $\left(x \in \left(\bigcap_{\lambda\in\Lambda} E_\lambda\right)^c\right) \Leftrightarrow \left(x \notin \bigcap_{\lambda\in\Lambda} E_\lambda\right) \Leftrightarrow$

(there is $\lambda\in\Lambda$ s.t. $x \notin E_\lambda$) $\Leftrightarrow$ (there is $\lambda\in\Lambda$ s.t. $x \in E_\lambda^c$) $\Leftrightarrow$

$\Leftrightarrow \left(x \in \bigcup_{\lambda\in\Lambda} E_\lambda^c\right)$

(b) • Let F_i, $1 \le i \le n$ be closed sets, then F_i^c are open sets.

Set $\left(\bigcup_{i=1}^n F_i\right)^c = \bigcap_{i=1}^n F_i^c$ is open, so $\bigcup_{i=1}^n F_i$ is closed.

• Let F_λ, $\lambda\in\Lambda$ be closed sets, then F_λ^c are open sets.

Set $\left(\bigcap_{\lambda\in\Lambda} F_\lambda\right)^c = \bigcup_{\lambda\in\Lambda} F_\lambda^c$ is open, so $\bigcap_{\lambda\in\Lambda} F_\lambda$ is closed.