Homework # 1 (Solutions)

1 (a) <u>Proof by contradiction.</u> Assume $\sqrt{15}$ is rational, i.e. $\sqrt{15} = \frac{m}{n}$, where $m \in \mathbb{N}$, $n \in \mathbb{Z}$, $n \neq 0$ and the fraction is <u>in its lowest form</u>, i.e. m and n do not have a common multiple.

Then $m^2 = 15 n^2$. This implies that m^2 is divisible by 3.

By considering 3 cases:

$m = 3k$, $m^2 = 9k^2$, $k \in \mathbb{N}$;

$m = 3k+1$, $m^2 = 9k^2 + 6k + 1$, $k \in \mathbb{N}$;

$m = 3k+2$, $m^2 = 9k^2 + 12k + 4$, $k \in \mathbb{N}$,

we see that m^2 is divisible by 3 if and only if m is divisible by 3.

Thus $m = 3k$, $k \in \mathbb{N}$ $\Rightarrow$ $9k^2 = 15n^2 \Rightarrow$ $3k^2 = 5n^2 \Rightarrow$ n^2 is divisible by 3.

Thus $n = 3r$, $r \in \mathbb{Z}$, and the fraction $\frac{m}{n} = \frac{3k}{3r} = \frac{k}{r}$ is not in its lowest form, which is a contradiction.

(b) Proof by contradiction: assuming that $\sqrt{3} + \sqrt{5} = r$, where $r \in \mathbb{Q}$, we conclude that $(\sqrt{3} + \sqrt{5})^2 = 8 + 2\sqrt{15} = r^2 \in \mathbb{Q}$, and that $\sqrt{15} = \frac{r^2 - 8}{2} \in \mathbb{Q}$, which is a contradiction.

2(a) Proof by induction on n:

Checking Base step $n=1$: $a + aq = a\left(\frac{1-q^2}{1-q}\right) = a\frac{(1-q)(1+q)}{1-q}$

$= a(1+q)$ ✓

Induction step: Assume that the formula is true when $n=k$, i.e. $a + aq + \dots + aq^k = a\frac{1-q^{k+1}}{1-q}$

Then $a + aq + \dots + aq^k + aq^{k+1} = a\frac{1-q^{k+1}}{1-q} + aq^{k+1}$

$= a\frac{1-q^{k+1} + (1-q)q^{k+1}}{1-q} = a\frac{1-q^{k+2}}{1-q}$, and the formula is true when $n=k+1$.

2(b) Proof by induction on n:

Checking base step $n=1$: $1+x \geqslant 1+x$ for all $x \geqslant -1$ ✓.

Induction step: Assume that inequality is true when $n = k \in \mathbb{N}$:

$(1+x)^k \geqslant 1+kx$, $x \geqslant -1$

Since $x \geqslant -1$, $x+1 \geqslant 0$ and we can multiply both sides of $(1+x)^k \geqslant 1+kx$ by a nonnegative number $1+x$:

$(1+x)(1+x)^n \geqslant (1+x)(1+kx)$, i.e.

$(1+x)^{n+1} \geqslant 1 + x + kx + kx^2 \geqslant 1+(k+1)x$, $x \geqslant -1$.

Thus, inequality is valid for $n=k+1$.