Homework # 19 (Solutions)

1. Note that as stated the statement is false. For example, let $A = \{0\}$ and $a_n = 0 \ \forall n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} a_n = 0 = x$, yet $x = 0$ is not a limit point of A.

Correct statement includes an additional assumption that $a_n \neq x \ \forall n \in \mathbb{N}$.

Given this assumption, we check def. of limit point: $\forall \varepsilon > 0$, all but finitely many elts of (a_n) are in $V_\varepsilon(x)$, so $V_\varepsilon(x) \cap (A - \{x\}) \neq \emptyset$.

2. a) The limit points are $\frac{\sqrt{2}}{2}, 1, -\frac{\sqrt{2}}{2}, -1$.

If $a_n = \dfrac{n\sin\left(\frac{n\pi}{4}\right)}{n+1}$, then

$\lim\limits_{k\to\infty} a_{8k+1} = \frac{\sqrt{2}}{2}$, $\lim\limits_{k\to\infty} a_{8k+2} = 1$, $\lim\limits_{k\to\infty} a_{8k+5} = -\frac{\sqrt{2}}{2}$,

$\lim\limits_{k\to\infty} a_{8k+6} = -1$.

b) Yes, every element of A is isolated point.

c) $\overline{A} = A \cup \left\{\frac{\sqrt{2}}{2}, 1, -\frac{\sqrt{2}}{2}, -1\right\}$.

3. a) This set is not open since $V_\varepsilon(1)$ is not contained in the set $\forall \varepsilon > 0$.

It is closed since the only limit point 0 is in the set.

b) $\bigcup_{n=1}^{\infty} [1-1/n, +\infty) = [0, +\infty)$.

This set is not open since $V_\varepsilon(0) \not\subset [0, +\infty)$ $\forall \varepsilon > 0$

The set is closed since any convergent sequence $(a_n) \subset [0, +\infty)$ converges to an elt of $[0, +\infty)$

c) $\mathbb{R}$ is both closed and open.

- Any limit point of $\mathbb{R}$ is a real number and thus is in $\mathbb{R}$, so $\mathbb{R}$ is closed.
- $\forall x \in \mathbb{R}$, $V_1(x) \subset \mathbb{R}$, so $\mathbb{R}$ is also open.

d) This set is not open, since $V_\varepsilon(0)$ is not a subset of $\mathbb{Q} \cap [0,1]$ $\forall \varepsilon > 0$.

It is also not closed since $\frac{\sqrt{2}}{2}$ is a limit point of the set that is not in the set.

4a) $S_K = \sum_{n=1}^{K} (\sqrt{n+1} - \sqrt{n}) = \sqrt{K+1} - 1$

$\lim_{K \to \infty} S_K = +\infty$, so series diverges by definition.

4b) Observe that $e^{-n^2} \le e^{-n}$ since $n^2 \geq n$.

$\sum_{n=1}^{\infty} e^{-n}$ is geometric with $q = e^{-1} < 1 \Rightarrow$

$\sum_{n=1}^{\infty} e^{-n^2} < \sum_{n=1}^{\infty} e^{-n}$

series converges absolutely

4c) $\sin\left(\frac{1}{n}\right) > 0$, $\sin\left(\frac{1}{n}\right) > \sin\left(\frac{1}{n+1}\right)$ $\forall n \in \mathbb{N}$ since $\sin(x)$ increases on $[0, \frac{\pi}{2}]$.

$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$ converges by Alt. series test.

Since $\lim_{x\to 0} \frac{\sin x}{x} = 1 \Rightarrow \lim_{x\to 0} \frac{\sin\left(\frac{1}{n}\right)}{1/n} = 1$.

Thus for n large enough $\sin(1/n) > \frac{1}{2}\cdot\frac{1}{n}$

Since $\sum_{K=N}^{\infty} \frac{1}{2}\cdot\frac{1}{n}$ diverges, $\sum_{k=1}^{\infty} \sin\left(\frac{1}{n}\right)$ also diverges by comparison test. Conditionally convergent

4d) $\sum_{n=1}^{\infty} \frac{2^n+4^n}{3^n+5^n} < \sum_{n=1}^{\infty} \frac{2^n+4^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$

Abs. convergent by comparison with two convergent geometric series.

e) Ratio test $\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left| \frac{(n+1)^{1000}}{n^{1000}} \cdot \frac{(1.01)^n}{(1.01)^{n+1}} \right|$

$= \lim \left(1+\frac{1}{n}\right)^{1000} \cdot \frac{1}{1.01} = \frac{1}{1.01} < 1 \Rightarrow$

converges absolutely.

3.2.6) a) False. $(-\infty, \sqrt{2}) \cup (\sqrt{2}, +\infty)$ is open and contains $\mathbb{Q}$.

b) False. Sets $A_n = [n, +\infty)$ are nonempty closed and nested, yet $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

c) True. Suppose $U \neq \emptyset$ is open and $x \in U$.
By def. of open, $\exists \, \varepsilon > 0$ s.t. $V_\varepsilon(x) \subset U$.

By density of rationals, $\exists \, r \in \mathbb{Q}$, $x < r < x + \varepsilon \Rightarrow$
$r \in U$

d) False, set $F = \{\sqrt{2}\} \cup \{\sqrt{2} + \frac{1}{n} \mid n \in \mathbb{N}\}$
is closed, infinite, bounded, but contains no rationals.

e) True. $C := \bigcap_{n=1}^{\infty} C_n$, where each C_n is closed
as a finite union of closed intervals.
Since C is an intersection of closed sets, it is closed