Homework #18 (Solutions)

1 (a) Let $A_n = \sum_{k=1}^{n} a_k$ and $B_n = \sum_{k=1}^{n} b_k$.

Since $0 \le a_k \le b_k \ \forall k$, sequences (A_n) and (B_n) are increasing and $A_n \le B_n \ \forall n \in \mathbb{N}$.
$\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow (B_n)$ converges, then (B_n) is bounded $\Rightarrow (A_n)$ is bounded $\Rightarrow (A_n)$ converges by the Monotone Convergence Theorem. $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

(b) If (A_n) diverges, then (B_n) also have to diverge, since if instead (B_n) converges, (A_n) would have to converge too by (a).

2. We will check Cauchy Criterion for $\sum_{n=1}^{\infty} a_n$.

Given $\varepsilon > 0$, since $\sum_{n=1}^{\infty} |a_n|$ converges, by Cauchy criterion applied to this series $\exists N$ s.t. $\forall n \ge N$ and $k \in \mathbb{N}$, $\big| |a_{n+k}| + \dots + |a_n| \big| < \varepsilon$
Then $|a_{n+k} + \dots + a_n| \le |a_{n+k}| + \dots + |a_n| < \varepsilon$.
triangle inequality.
Thus $\sum_{n=1}^{\infty} a_n$ converges by Cauchy Criterion applied to this series.

3) #2.7.9 (a) Choose r' such that $r < r' < 1$.
Since $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$, for $\varepsilon = r' - r$

there exists $N(\varepsilon)$, s.t. $\forall n \ge N$ $\left| \left|\frac{a_{n+1}}{a_n}\right| - r \right| < \varepsilon \Rightarrow$

$\left|\frac{a_{n+1}}{a_n}\right| - r < \varepsilon \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| < \varepsilon + r = r' - r + r = r'$

I.e. $|a_{n+1}| \le r' |a_n| \ \forall n \ge N$.

b) $|a_N| \sum_{n=0}^{\infty} (r')^n$ converges to $|a_N| \cdot \frac{1}{1-r'}$

as a geometric series with $|q| = |r'| < 1$.

c) We will apply Cauchy criterion to the series $\sum_{n=1}^{\infty} |a_n|$.

Observe that if $n \geq N$ and $k \in \mathbb{N}$, then
$|a_{n+k} + a_{n+k-1} + \ldots + a_n| \leq$
$\leq |a_{n+k}| + |a_{n+k-1}| + \ldots + |a_{n+1}| + |a_n|$

$\leq (r')^k |a_n| + (r')^{k-1} |a_n| + \ldots + r'|a_n| + |a_n|$

$\leq |a_n| \cdot \sum_{i=0}^{k} (r')^i \leq |a_n| \cdot \sum_{i=0}^{\infty} (r')^i = |a_n| \cdot \frac{1}{1-r'}$

$\leq (r')^{n-N} |a_N| \cdot \frac{1}{1-r'}$

Now given $\varepsilon > 0$, choose N_1 so that
$\forall n \geq N_1$, $(r')^{n-N} |a_N| \cdot \frac{1}{1-r'} < \varepsilon$
(i.e. $N_1 > N + \left(\ln \frac{\varepsilon \cdot (1-r')}{|a_N|}\right) / \ln(r')$)

Let $N_2 = \max\{N, N_1\}$. Then the calculation above showes that $\forall n \geq N_2$ and $\forall k \in \mathbb{N}$

$|a_{n+k} + \ldots + a_n| < \varepsilon$.

4 (a) Converges by comparison test:

$$\sum_{n=1}^{\infty} \frac{3n}{n^3+1} < \sum_{n=1}^{\infty} \frac{3}{n^2}$$ converges by p-series test.

(b) $a_n = \frac{\log n}{n} > 0$, if $f(x) = \frac{\log x}{x}$, then

$$f'(x) = \frac{1}{\ln 10 \cdot x^2} - \frac{\log x}{x^2} < 0 \text{ when } x \geq 10$$

$\Rightarrow a_n$ is decreasing for $n \geq 10$

$$\lim_{n\to\infty} \frac{\log n}{n} = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n \log n}{n} \text{ converges}$$

by Alt. series test.

(c) Since $\sqrt[n]{n} \leq 2$ ($\Leftrightarrow n \leq 2^n$) $\Rightarrow$

$$\frac{1}{\sqrt[n]{n}} \geq \frac{1}{2} \Rightarrow \lim_{n\to\infty} \frac{(-1)^n}{\sqrt[n]{n}} \text{ does not exist}$$

and $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}$ diverges by divergence test.

(d) $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1,$$

so $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges by ratio test.