Homework 17 (Solutions)

1(a) Choose $\varepsilon = \frac{1}{2}$ and let N be arbitrary, let $m = 3 + 12N$ and $n = 6 + 12N$, then $m, n \geq N$ and $|a_n - a_m| = |0 - 1| = 1 > \frac{1}{2}$.

(b) Given $\varepsilon > 0$, choose $N > \sqrt{\frac{2}{\varepsilon}}$, then $\forall n \geq N$ and $\forall k \in \mathbb{N}$

$$|b_{n+k} - b_n| = \left| \frac{(-1)^{n+k}}{(n+k)^2} + \frac{(-1)^n}{n^2} \right| \leq \frac{1}{(n+k)^2} + \frac{1}{n^2}$$

$$\leq \frac{2}{n^2} \leq \frac{2}{N^2} < \frac{2}{\sqrt{2/\varepsilon}} = \varepsilon.$$

2. Observe that

$$|c_{n+k} - c_n| = |c_{n+k} - c_{n+k-1} + c_{n+k-1} - c_{n+k-2} + c_{n+k-2} - c_{n+k-3} + \dots + c_{n+1} - c_n| \leq$$

$$\leq |c_{n+k} - c_{n+k-1}| + |c_{n+k-1} - c_{n+k-2}| + \dots + |c_{n+1} - c_n|$$

$$\leq \left(\tfrac{2}{3}\right)^{n+k-1} + \left(\tfrac{2}{3}\right)^{n+k-2} + \dots + \left(\tfrac{2}{3}\right)^{n}$$

$$= \left(\tfrac{2}{3}\right)^n \left[\left(\tfrac{2}{3}\right)^{k-1} + \left(\tfrac{2}{3}\right)^{k-2} + \dots + 1 \right]$$

$$= \left(\tfrac{2}{3}\right)^n \cdot \frac{1 - \left(\frac{2}{3}\right)^k}{1 - \frac{2}{3}} \leq \left(\tfrac{2}{3}\right)^n \cdot 3$$

Given $\varepsilon > 0$, choose N such that $\left(\frac{2}{3}\right)^N \cdot \frac{1}{3} < \varepsilon$ $\left(\text{i.e. } N > \frac{\log(\varepsilon/3)}{\log(2/3)}\right)$

Then $\forall n \geq N$ and $\forall k \in \mathbb{N}$

$$|c_{n+k} - c_n| \leq \left(\tfrac{2}{3}\right)^n \cdot 3 \leq \left(\tfrac{2}{3}\right)^N \cdot 3 < \varepsilon.$$

2.6.2 a) Sequence $a_n = \frac{(-1)^n}{n}$ is Cauchy, since (a_n) converges to 0. (a_n) is not monoton

b) Impossible: If (a_n) has an unbounded subsequence, (a_n) itself is unbounded, then (a_n) can't be Cauchy, since all Cauchy sequences are bounded

c) Cauchy subsequence will be convergent. Monotone unbounded sequences can't have convergent subsequences (see 2.5.2 (d))

d) This is possible. Let $a_n = \begin{cases} 1, & n = 2k \\ n, & n = 2k+1 \end{cases}$.
Then (a_{2k}) is Cauchy, but (a_n) is unbounded

2.6.3 b) Since both (x_n) and (y_n) are Cauchy, they are bounded. I.e. $\exists A, B$ s.t. $|x_n| \le A$ and $|y_n| \le B$ $\forall n \in \mathbb{N}$.
Given $\varepsilon > 0$, choose N_1 s.t. $\forall m, n \geq N_1$, $|x_m - x_n| < \frac{\varepsilon}{2B}$, and choose N_2 s.t. $\forall m, n \geq N_2$, $|y_m - y_n| < \frac{\varepsilon}{2A}$.
Let $N = \max\{N_1, N_2\}$
Then $\forall m, n \geq N$

$$|x_m y_m - x_n y_n| = |x_m y_m - x_n \cdot y_m + x_n \cdot y_m - x_n y_n|$$
$$\le |y_m| \cdot |x_m - x_n| + |x_n| |y_m - y_n|$$
$$\le B \cdot |x_m - x_n| + A \cdot |y_m - y_n| < B \cdot \frac{\varepsilon}{2B} + A \cdot \frac{\varepsilon}{2A} = \varepsilon.$$

2.7.1(c)

$S_{2n+1} = a_1 - a_2 + a_3 - a_4 + \ldots + a_{2n+1} = S_{2n-1} - a_{2n} + a_{2n+1}$

Since a_n is decreasing, $-a_{2n} + a_{2n+1} \leq 0 \Rightarrow$
$S_{2n+1} \leq S_{2n-1}$ $\forall n \in \mathbb{N}$.

$S_{2n+1} = (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2n-1} - a_{2n}) + a_{2n+1} \geq 0$
(since all the terms in paranthesis are non-negative).

Sequence (S_{2n+1}) is decreasing and bounded from below by zero $\Rightarrow$ $\lim S_{2n+1}$ exists.
Let $\lim(S_{2n+1}) = L$.

In a similar way, one can show that sequence (S_{2n}) is decreasing and is bounded from above by a_1, and thus it also has a limit.

More directly, $S_{2n} = S_{2n+1} - a_{2n+1} \Rightarrow$
$\lim S_{2n} = \lim S_{2n+1} - \lim a_{2n+1} = L - 0$
(by Alg. limit theorem).

To show that $\lim S_n = L$ one can argue as follows: Given $\varepsilon > 0$, choose N_1 s.t. $\forall k \geq N_1$
$|S_{2k+1} - L| < \varepsilon$ and choose N_2 s.t. $\forall \ell \geq N_2$
$|S_{2\ell} - L| < \varepsilon$. Let $N = 2 \cdot \max\{N_1, N_2\} + 1$.
Then $\forall n \geq N$, $n = 2k+1$ with $k \geq N_1$ or
$n = 2\ell$ with $\ell \geq N_2$, so
$|S_n - L| < \varepsilon$.