Homework 15 (Solutions)

1. a) Let $n_k = 6 + 8k$, then $a_{n_k} = -\frac{1}{6+8k}$

Subsequence $\left(-\frac{1}{6+8k}\right)_{k=1}^{\infty}$ is $\underbrace{\text{decreasing}}^{\text{strictly}}$.

b) Let $n_k = 2+8k$, then $a_{n_k} = \frac{1}{2+8k}$ and subsequence $\left(\frac{1}{2+8k}\right)_{k=1}^{\infty}$ is strictly decreasing.

c) Let $n_k = 4k$, then $a_{n_k} = 0$ and subsequence $(0)_{k=1}^{\infty}$ is constant.

d) Let $n_k = (2+4k)$, then $\sin\left(\frac{(2+4k)\pi}{4}\right) = (-1)^{k+1}$ and $a_{n_k} = \frac{(-1)^{k+1}}{2+4k}$ changes sign from a_{n_k} to $a_{n_{k+1}}$.

$\boxed{\# \ 2.5.1}$ a) Not possible. A subsequence is also a sequence on its own right. Since it is bounded, it has a convergent subsequence by Bolzano-Weierstrass. This subsequence is also a subsequence of the original sequence.

b)

Let $a_n = \frac{1}{2} + \frac{(-1)^n n}{2n+1}$, then

$$a_{2k} = \frac{1}{2} + \frac{2k}{4k+1} = \frac{8k+1}{2(4k+1)}, \quad \lim_{k\to\infty} a_{2k} = 1.$$

$$a_{2k+1} = \frac{1}{2} - \frac{2k+1}{2(2k+1)+1} = \frac{1}{8k+6}, \quad \lim_{k\to\infty} a_{2k+1} = 0.$$

c) An example of such sequence is

$1, 1, 1/2, 1, 1/2, 1/3, 1, 1/2, 1/3, 1/4, \ldots, 1, 1/2, 1/3, \ldots, 1/n, \ldots$

This sequence contains infinitely many elements equal to $1/n$ for each n. So it has a constant subsequence convergent to $1/n$ $\forall n$.

Another example of such a sequence is all rational numbers.

(d) Not possible. Such a sequence (x_n) must have a subsequence converging to 0.
To see this observe that intervals $I_k = [1/k - \frac{1}{2^k}, \frac{1}{k} + \frac{1}{2^k}]$ must contain infinitely many elements of of (x_n), since $\forall k$ there is a subsequence of (x_n) convergent to $1/k$.
Now choose $x_{n_1} \in I_1$, $x_{n_2} \in I_2 - \{x_1, \ldots, x_{n_1}\}, \ldots$
$x_{n_k} \in I_k - \{x_1, \ldots, x_{n_{k-1}}\}, \ldots$

Then $\frac{1}{k} - \frac{1}{2^k} \leq x_{n_k} \leq 1/k + \frac{1}{2^k}$ $\forall k \in \mathbb{N}$.

By order limit theorem $\lim_{k\to\infty} x_{n_k} = 0$ since $\lim_{k\to\infty} (1/k \pm \frac{1}{2^k}) = 0$.

$\boxed{\#2.5.5}$ If (a_n) is bounded and diverges, we proved in 2.5.2 (c) that (a_n) must have two subsequences converging to different limits. Since all convergent subsequences of (a_n) converge to the same limit, (a_n) must to converge itself.

2.5.2

(a) True. Subsequence (y_k), where $y_k = x_{k+1}$ is a proper subsequence of (x_k)
If $\lim y_k = L$, then $\lim x_k = \lim y_{k-1} = L$, so (x_k) also converges.

(b) True. If (x_n) converges, then any subsequence of (x_n) converges to the same limit. So convergent sequences cannot have divergent subsequences.

(c) True. Since (x_n) is bounded, it has a convergent subsequence (x_{n_k}). Let $\lim_{k \to \infty} x_{n_k} = L$. Since (x_n) does not converge to L,

$\exists \varepsilon > 0$ such that there are infinitely many elements of (x_n) outside of $V_\varepsilon(L) = (L-\varepsilon, L+\varepsilon)$
Thus we can choose a subsequence (y_m) of (x_n) so that $y_m \notin V_\varepsilon(L)$ $\forall m$.
Since (y_m) is bounded, it has convergent subsequence (y_{m_ℓ}) that cannot converge to L.
Thus, (x_{n_k}) and (y_{m_ℓ}) are two subsequences of (x_n) that have the same limit.

(d) True, Assume without loss of generality that (x_n) is increasing.
Let (x_n) has a subsequence (x_{n_k}) that converges to x.
Then given $\varepsilon > 0$ $\exists N(\varepsilon)$ s.t $\forall k \geq N(\varepsilon)$ $x_{n_k} \in V_\varepsilon(x)$. Since (x_n) is increasing, if $m \geq n_{N(\varepsilon)}$, then $x_m \geq x_{n_{N(\varepsilon)}}$, so $x_m \in V_\varepsilon(x)$.