Homework # 14 (Solutions)

#1 Let $A_n = \{ a_k \mid k \geq n \}$.

a) $A_n = \{ (-1)^n, (-1)^{n+1}, (-1)^{n+2}, \dots \}$

$y_n = \sup(A_n) = 1$, $z_n = \inf(A_n) = -1$

$\limsup a_n = 1$, $\liminf a_n = -1$

b) $a_n = \sin\left(\frac{n\pi}{6}\right)$, $A = \left\{ \frac{1}{2}, \frac{\sqrt{3}}{2}, 1, \sqrt{3}/2, \frac{1}{2}, 0, -1/2, -\frac{\sqrt{3}}{2}, -1, -\sqrt{3}/2, -1/2, 0, \text{pattern repeats} \dots \right\}$

$y_n = \sup(A_n) = 1$, $z_n = \inf(A_n) = -1$

$\limsup a_n = 1$, $\liminf a_n = -1$

c) $A = \{ -1/2, 2/3, -3/4, 4/5, \dots \}$

$y_n = 1$, $z_n = -1$, $\limsup(a_n) = 1$, $\liminf(a_n) = -1$

#2. Observe that if $p > 0$ $n(\ln n)^p < (n+1)(\ln(n+1))^p$

$\Rightarrow$ sequence $1/n(\ln n)^p$ is decreasing.

Applying Cauchy condensation test

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges/diverges if and only if

$$\sum_{k=1}^{\infty} \frac{2^k}{2^k(\ln(2^k))^p} = \sum_{k=1}^{\infty} \frac{1}{(\ln 2)^p \cdot k^p} \text{ converges/diverges.}$$

Thus $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is $\begin{cases} \text{converging} & p > 1 \\ \text{diverging} & 0 < p \leq 1 \end{cases}$

If $p \leq 0$, $n(\ln n)^p \leq n$, so

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \geq \sum_{n=2}^{\infty} \frac{1}{n} = +\infty$, so the series

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ diverges for $p \geq 0$.

#3) a) $S_n = \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \ldots + \left(\frac{2}{3}\right)^n$

$$= \frac{(2/3) - (2/3)^{n+1}}{1 - 2/3}$$

$\boxed{\text{Series converges to } 2.}$

$$\lim_{n\to\infty} S_n = \frac{2/3}{1/3} = 2 \quad \text{since} \quad \lim_{n\to\infty} (2/3)^n = 0.$$

b) $S_n = \sum_{k=1}^{n} \frac{\ln(k+1)}{\ln(k)} = (\cancel{\ln 2} - \cancel{\ln 1}) + (\ln 3 - \cancel{\ln 2}) + \ldots$

$+ (\ln(n+1) - \cancel{\ln(n)}) = \ln(n+1)$

$\lim_{n\to\infty} S_n = +\infty$, so series diverges to $+\infty$.

c) $\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$

$$S_n = \frac{1}{2}\left[\left(1 - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{5}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{6}}\right) + \ldots\right.$$

$$\left. + \left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)\right] = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$$

$\lim_{n\to\infty} S_n = 3/4$, so series converges to $3/4$.

4 a) Observe that since $\left(a - \frac{2}{a}\right)^2 \geq 0 \Rightarrow$

$a^2 - 4 + \frac{4}{a^2} \geq 0 \Rightarrow a^2 + \frac{4}{a^2} \geq 4$.

Then $(x_n)^2 = \frac{1}{4}\left(x_{n-1}^2 + 4 + \frac{4}{x_{n-1}^2}\right) \geq \frac{1}{4}(4+4) = 2.$

$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) - x_n = \frac{1}{2}\left(\frac{2}{x_n} - x_n\right)$

$= \frac{1}{2x_n}(2 - x_n^2) \geq 0$

Thus (x_n) is a decreasing sequence.

It is bounded below by 0 ($x_1 > 0$, $x_n > 0 \Rightarrow$

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) > 0$

By the Monotone Convergence Theorem (x_n) converges to some number L.

Taking limit on both sides of

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ we get

$L = \frac{1}{2}\left(L + 2/L\right) \Rightarrow L = \pm\sqrt{2}$. Since

$x_n > 0 \Rightarrow L = \sqrt{2}$.