Homework # 13 (Solutions)

#1) Preliminary work: Given $M > 0$, we want to have $\frac{n^2-3}{n+4} > M$.

Observe that for $n \geq 4$, $n+4 \leq 2n$ and $n^2 - 3 > \frac{1}{2}n^2 \Rightarrow$

$$\frac{n^2-3}{n+4} \geq \frac{\frac{1}{2}n^2}{2n} = \frac{n}{4} > M,$$ $n > 4M$ will work.

Proof: Given $M > 0$, choose an integer $N > 4M$, then $\forall n \geq N$,

$$\frac{n^2-3}{n+4} \geq \frac{n}{4} \geq \frac{4N}{4} \geq \frac{4M}{4} = M.$$

Thus,

$$\lim_{n\to\infty} \frac{n^2-3}{n+4} = +\infty.$$

#2)

$$\lim_{n\to\infty} \frac{(2n+1)^{10} \cdot (3n-1)^{20}}{(n+7)^{30}}$$

by Thm. 2.3.3(iii)

$$= \lim_{n\to\infty} \left(\frac{2n+1}{n+7}\right)^{10} \cdot \left(\frac{3n-1}{n+7}\right)^{20} = L^{10} \cdot M^{20} = 2^{10} \cdot 3^{20}$$

where

Thm 2.3.3(iv)

$$L = \lim_{n\to\infty} \frac{2n+1}{n+7} = \lim_{n\to\infty} \frac{2 + \frac{1}{n}}{1 + \frac{7}{n}} = \frac{\lim_{n\to\infty} (2 + \frac{1}{n})}{\lim_{n\to\infty} (1 + \frac{7}{n})}$$

$$= \frac{2}{1}.$$

Similarly, $M = \lim_{n\to\infty} \frac{3n-1}{n+7} = \lim_{n\to\infty} \frac{3 - \frac{1}{n}}{1 + \frac{7}{n}} = 3$

#3) a) $a_1 = \sqrt{2}$, $a_2 = \sqrt{\sqrt{2}+2}$,

$a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$, $a_4 = \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$

b) Base step: $a_1 = \sqrt{2} < 2$ ✓

Induction step: Suppose $a_n \leq 2$

$\Rightarrow a_{n+1} = \sqrt{a_n+2} \leq \sqrt{2+2} = 2.$

c) Base step: $a_1 = \sqrt{2} \leq a_2 = \sqrt{2+\sqrt{2}}$ ✓

Induction step:

Assume that $a_{n-1} \leq a_n$,

Then $a_n = \sqrt{a_{n-1}+2} \leq \sqrt{a_n+2} = a_{n+1}$.

d) Sequence (a_n) is increasing and bounded above, so by Monotone Convergence thm it has a limit. Let $\lim a_n = L$.

Then

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{a_n+2} \overset{\text{by exercise 2.3.1}}{=} \sqrt{\lim_{n\to\infty} a_n + 2} = \sqrt{L+2}$$

Thus $L = \sqrt{L+2}$ or $L^2 - L - 2 = 0$

Solving for L we get $L_1 = 2$, $L_2 = -1$.

L_2 is not possible, since $L \geq 0$.

Thus

$$\lim_{n\to\infty} a_n = 2.$$

#2.4.7 (a) Let $A_n = \{a_k \mid k \geq n\} = \{a_n, a_{n+1}, \dots\}$.
Observe that $A_1 \supseteq A_2 \supseteq \dots \supseteq A_n \supseteq A_{n+1} \supseteq \dots$

Observe that each A_n is bounded, since we assumed that the set $A_1 = \{a_1, a_2, \dots\}$ is bounded, thus $y_n = \sup A_n$ exists $\forall n$, by the Axiom of completeness.
Also, since $A_n \supseteq A_{n+1}$, any upper bound of A_n is also an upper bound of A_{n+1} $\Rightarrow$
$y_1 \geq y_2 \geq \dots \geq y_n \geq y_{n+1} \geq \dots K$, where K is any lower bound of A_1.
Thus, sequence (y_n) is decreasing and is bounded from below. By Monotone convergence theorem it has a limit denoted $\limsup (a_n) := \lim y_n$.

(b) Similarly, let $z_n = \inf(A_n)$. z_n exists, since each set A_n is bounded from below.
Since any lower bound of A_n is also a lower bound of A_{n+1} $\forall n \in \mathbb{N}$, we have.
$z_1 \leq z_2 \leq \dots \leq z_n \leq z_{n+1} \leq \dots \leq M$,
where M is any upper bound of A_1.
By the Monotone convergence theorem (z_n) has a limit denoted $\liminf(a_n) := \lim z_n$.
(c) Since $z_n = \inf\{a_k \mid k \geq n\} \leq a_n \leq \sup\{a_k \mid k \geq n\} = y_n$, by the Order limit theorem

$$L = \lim_{n\to\infty} z_n \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} y_n = L$$

$\Rightarrow$ If $\lim z_n = \lim y_n = L$, then (a_n) also converges to L.

Now suppose $\lim a_n = L$.

Then for any given $\varepsilon > 0$, there exists $N(\varepsilon)$ s.t. $\forall\, n \geq N(\varepsilon)$ $|a_n - L| < \varepsilon$,

i.e. $\forall\, n \geq N(\varepsilon)$ $L - \varepsilon < a_n < \varepsilon + L$

This implies that $\varepsilon + L$ is an upper bound of the set $\{a_k \mid k \geq n\}$ and $L - \varepsilon$ is its lower bound $\Rightarrow$

$L - \varepsilon < z_n \leq y_n \leq L + \varepsilon$ $\forall\, n \geq N(\varepsilon)$.

Thus, the same $N(\varepsilon)$ works in proving that $\lim y_n = L$ and $\lim z_n = L$.