Homework #12 (Solutions)

1. Suppose $\underbrace{(\sin(\frac{n\pi}{2}))_{n=1}^{\infty}}_{=(a_n)}$ does have a limit ℓ.
Then for $\varepsilon = 1/4$ one can find $N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - \ell| < 1/4$ $\Rightarrow$
$\forall n, m \geq N$ $|a_n - a_m| = |a_n - \ell + \ell - a_m| \leq$
$\leq |a_n - \ell| + |a_m - \ell| < 1/4 + 1/4 = 1/2$.
On the other hand, if $n = 2N$, then $a_n = 0$ and if $m = 2N+1$, then $a_m = 1$, and $|a_n - a_m| = 1 > 1/2$. This is a contradiction with the existence of the limit.

#2 a) Suppose $\lim_{n\to\infty} b_n = b$ and $b \neq 0$.

Then choose $\varepsilon = \frac{|b|}{2} > 0$ and find $N(\varepsilon)$ s.t.
$\forall n \geq N(\frac{|b|}{2}) = N_1$, $|b_n - b| < \varepsilon$.
Then $\forall n \geq N_1$, $|b| - |b_n| \leq |b_n - b|$ $\Rightarrow$
$$|b_n| \geq |b| - |b_n - b| > |b| - \varepsilon = \frac{|b|}{2}.$$

b) Given $\varepsilon > 0$, let $\varepsilon_2 = \varepsilon \cdot |b|^2/2$ and since $\lim_{n\to\infty} b_n = b$, one can find $N_2(\varepsilon)$ such that $|b_n - b| < \varepsilon \cdot |b|^2/2$.

Then take $N(\varepsilon) = \max\{N_1, N_2(\varepsilon)\}$.
For any $n \geq N(\varepsilon)$
$$\left|\frac{1}{b} - \frac{1}{b_n}\right| = \frac{|b_n - b|}{|b| \cdot |b_n|} \overset{(|b_n| > |b|/2)}{<} \frac{|b_n - b|}{|b| \cdot |b|/2} < \frac{\varepsilon \cdot |b|^2/2}{|b|^2/2} = \varepsilon.$$

(c) We already proved in class that
$\lim a_n/b_n \overset{\text{in class}}{=} \lim a_n \cdot \lim 1/b_n = a \cdot 1/b$

#2.3.7 (a) $x_n = (-1)^n$, $y_n = -(-1)^n$ both diverge, $(x_n + y_n) = (0) \leftarrow$ converges.

(b) Let $z_n = x_n + y_n$ $\forall n$.
then $z_n - x_n = y_n$. If both (z_n) and (x_n) converge, then (y_n) also conveges by alg. limit theorem $\Rightarrow$ the request is impossible.

(c) Let $b_n = \frac{1}{n} \neq 0$ $\forall n \in \mathbb{N}$, $\frac{1}{b_n} = n$ and sequence $(n)_{n=1}^{\infty}$ diverges.

(d) Not possible. Since (b_n) is convergent, (b_n) is bounded, i.e. $\exists M_1$ s.t. $|b_n| \leq M_1$ $\forall n \in M_1$.
If $(a_n - b_n)$ is also bounded, i.e. $\exists M_2$ s.t. $|a_n - b_n| \leq M_2$ $\forall n \in \mathbb{N}$, then (a_n) is also bounded:

$$|a_n| = |a_n - b_n + b_n| \leq |a_n - b_n| + |b_n| \leq M_2 + M_1.$$

(e) Let $a_n = \frac{1}{n^2}$, $b_n = n$, $a_n \cdot b_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$.
Then both (a_n) and $(a_n b_n)$ converge to 0 and (b_n) diverges.