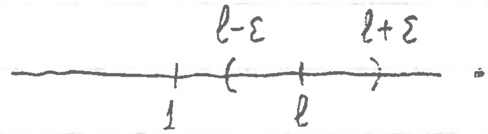
Hwk # 11 (Solutions)

#2.2.4. (a) $a_n = (-1)^n$ (proved that a_n does not converge to 1 in class)

(b) This is impossible If $\lim_{n\to\infty} a_n = \ell$, $\ell \neq 1$, then all but finitely many elements of this sequence are contained in $V(\varepsilon) = (\ell - \varepsilon, \ell + \varepsilon)$ with $\varepsilon = |\ell - 1|/2$. Since $1 \notin V(\varepsilon)$, such a sequence cannot contain infinite number of 1's.



(c) An example of such a sequence is

$$b_n = \begin{cases} k, & \text{if } n = 2^k \\ 1, & \text{if } n \neq 2^k \end{cases}, \quad k = 1, 2, \ldots$$

This sequence is not bounded, so it diverges. For each $k \in \mathbb{N}$, there is a $2^k - 2^{k-1} - 1 = 2^{k-1} - 1$ block of 1's.

#2.2.7. (a) Frequently, but not eventually: given any $N \in \mathbb{N}$, $a_{2N} = 1 \in \{1\}$; since $a_{2N+1} \notin \{1\}$ and since $2N+1 > N$, $(a_n)_{n=1}^{\infty}$ is not eventually in $\{1\}$.

(b) Eventually $\Rightarrow$ frequently. If (a_n) is in A eventually $\Rightarrow \exists N_1$ such that $\forall n \geq N_1$, $a_n \in A$ $\Rightarrow \forall N \; \exists n_1 \geq N$ (namely $n_1 = \max\{N, N_1\}$) such that $a_{n_1} \in A$

(c) $\left(\lim_{n\to\infty} a_n = L\right) \Longleftrightarrow \left(\forall \varepsilon > 0, (a_n) \text{ is eventually in } V_\varepsilon(L)\right)$

(d) (x_n) will be in A frequently:

if not, then $\exists N$ s.t. $\forall n \geq N$, $x_n \notin A$
$\Rightarrow \forall n \geq N$, $x_n \neq 2 \Rightarrow (x_n)$ has only finite number of 2's, which is a contradiction.

Taking $x_n = (-1)^n \cdot 2$, we see that (x_n) has infinite number of 2's and it is not eventually in $(1.9, 2.1)$

$\boxed{\#2.3.1}$ (a) Since $(x_n) \to 0$, for $\varepsilon_1 = \varepsilon^2$ $\exists N_1(\varepsilon_1)$ s.t $\forall n \geq N_1(\varepsilon_1)$, $|x_n - 0| < \varepsilon_1$.

Now, given any $\varepsilon > 0$, choose $N(\varepsilon) = N_1(\varepsilon_1)$. Then $\forall n \geq N(\varepsilon_1)$, $|x_n| < \varepsilon_1 \Rightarrow |\sqrt{x_n} - 0| < \varepsilon$.

(b) From the order limit theorem $\Rightarrow x \geq 0$. If $x = 0$, see the proof in (a). If $x > 0$, since $(x_n) \to x$, given $\varepsilon_1 = \varepsilon\sqrt{x}$ one can choose $N_1(\varepsilon_1)$ s.t $\forall n \geq N_1(\varepsilon_1)$, $|x_n - x| < \varepsilon_1$.

Now, given $\varepsilon > 0$, choose $N(\varepsilon) = N_1(\varepsilon_1)$.

Then $\forall n \geq N(\varepsilon)$, $|\sqrt{x_n} - \sqrt{x}| = \left| \dfrac{x_n - x}{\sqrt{x_n} + \sqrt{x}} \right|$

$$\leq \frac{|x_n - x|}{\sqrt{x}} < \frac{\varepsilon_1}{\sqrt{x}} = \frac{\varepsilon\sqrt{x}}{\sqrt{x}} = \varepsilon$$

#2.3.3. Since $\lim_{n\to\infty} x_n = l$ and $\lim_{n\to\infty} z_n = l$,

given ε one can choose $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$, s.t. $\forall n \geq N_1$, $|x_n - l| < \varepsilon$, i.e. $l - \varepsilon < x_n < l + \varepsilon$; s.t. $\forall n \geq N_2$, $l - \varepsilon < z_n < l + \varepsilon$. Then $\forall n \geq N = \max\{N_1, N_2\}$, $l - \varepsilon < x_n \leq y_n \leq z_n < l + \varepsilon \Rightarrow$ $|y_n - l| < \varepsilon$.