Homework #10 (Solutions)

#1. We will show that $\bigcup_{n=1}^{\infty} [1/n, n] = (0, +\infty)$.

- Observe that $[1/n, n] \subset (0, +\infty)$ $\forall n \in \mathbb{N}$. Thus $\bigcup_{n=1}^{\infty} [1/n, n] \subseteq (0, +\infty)$.
- Let $x \in (0, +\infty)$, by Archimedian Principle $\exists\, n_1, n_2 \in \mathbb{N}$ s.t. $\frac{1}{n_1} < x < n_2$. Let $n = \max\{n_1, n_2\}$, then $x \in [1/n, n] \Rightarrow x \in \bigcup_{n=1}^{\infty} [1/n, n]$. Thus $(0, +\infty) \subseteq \bigcup_{n=1}^{\infty} [1/n, n]$.

#2. $\lim_{n \to \infty} \frac{2n^3}{n^3+4} = 2$. We need to solve

$$\left| \frac{2n^3}{n^3+1} - 2 \right| < \varepsilon \quad \text{for } n.$$

$$\| $$

$$\left| \frac{-8}{n^3+4} \right| = \frac{8}{n^3+4} < \varepsilon \Rightarrow n > \left(\frac{8}{\varepsilon} - 4 \right)^{1/3}.$$

If $\varepsilon = .1$, $n > \left(\frac{8}{.1} - 4 \right)^{1/3}$, $N(.1) = 5$

If $\varepsilon = .01$, $n > \left(\frac{8}{.01} - 4 \right)^{1/3}$, $N(.01) = 10$.

#3 a) We will prove that $\lim_{n \to \infty} \frac{4n-1}{n+5} = 4$.

Preliminary work:

$$\left| \frac{4n-1}{n+5} - 4 \right| = \left| \frac{-21}{n+5} \right| = \frac{21}{n+5} < \varepsilon \Rightarrow n > \frac{21}{\varepsilon} - 5.$$

Proof: Given $\varepsilon > 0$, choose integer $N(\varepsilon) > 21/\varepsilon - 5$.
Then $\forall n \geq N(\varepsilon)$ one has

$$\left| \frac{4n-1}{n+5} - 4 \right| = \frac{21}{n+5} \leq \frac{21}{N(\varepsilon)+5} < \frac{21}{21/\varepsilon - 5 + 5} = \varepsilon.$$

b) We will prove that
$$\lim_{n\to\infty} \frac{2n^3}{n^3-n} = 2.$$
Preliminary work: $\left|\frac{2n^3}{n^3-n} - 2\right| = \left|\frac{2n}{n^3-n}\right|$
$$= \frac{2}{n^2-1} < \varepsilon \implies n > \sqrt{2/\varepsilon + 1}$$
Proof: Given $\varepsilon > 0$, choose integer $N(\varepsilon) > \sqrt{2/\varepsilon + 1}$

Then for all $n \geq N(\varepsilon)$ we have
$$\left|\frac{2n^3}{n^3-n} - 2\right| = \frac{2}{n^2-1} \leq \frac{2}{(N(\varepsilon))^2 - 1} < \frac{2}{\left(\sqrt{2/\varepsilon+1}\right)^2 - 1} = \varepsilon.$$

c) We will prove that $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$.

Preliminary work:
$$\left|\sqrt{n+1} - \sqrt{n}\right| = \left|\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}} < \varepsilon$$
$$n > \left(1/\varepsilon\right)^2.$$

Proof: Given $\varepsilon > 0$, choose an integer $N(\varepsilon) > \left(\frac{1}{\varepsilon}\right)^2$.
Then $\forall n \geq N(\varepsilon)$ one has
$$\left|\sqrt{n+1} - \sqrt{n} - 0\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N(\varepsilon)}} < \frac{1}{\sqrt{(1/\varepsilon)^2}} = \varepsilon.$$

2.2.1 Here are two examples of "vercongent" sequences: $a_n = (-1)^n$ and $b_n = 1/n$.
Both sequences "vercoge" to 0. Since for $\varepsilon = 2$, $|a_n - 0| < 2$ and $|b_n - 0| < 2$ for all $n \in \mathbb{N}$.
$\{a_n\}_{n=1}^{\infty}$ is an example of "vercongent" sequence that is divergent. Sequence $\{a_n\}_{n=1}^{\infty}$ "vercoges" to 0 and ½ with the same $\varepsilon = 2$.

If $\{a_n\}_{n=1}^{\infty}$ is vercogent to a, then

$\exists \varepsilon$ s.t. $|a_n - L| < \varepsilon \quad \forall n \in \mathbb{N} \Rightarrow$

$-\varepsilon + L < a_n < \varepsilon + L \Rightarrow |a_n| < \max\{|L-\varepsilon|, |L+\varepsilon|\}$.

Thus verconsent sequences are bounded.

Similarly if $\{a_n\}_{n=1}^{\infty}$ is bounded by $M > 0$

$\Rightarrow \{a_n\}_{n=1}^{\infty}$ vercoges to 0 with $\varepsilon = M+1$.

Thus vercogent sequences are the same as bounded sequences.

2.2.6) Given $\varepsilon > 0$, since $\lim_{n\to\infty} a_n = a \Rightarrow$

$\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$, $|a_n - a| < \varepsilon/2$.

Similarly, if $\lim_{n\to\infty} a_n = b \Rightarrow \exists N_2 \in \mathbb{N}$ s.t.

$\forall n \geq N_2 \Rightarrow |a_n - b| < \varepsilon/2$.

Let $n \geq \max\{N_1, N_2\} \Rightarrow$

$|a-b| = |a - a_n + a_n - b| = |a - a_n| + |a_n - b|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Since $\varepsilon > 0$ were arbitrary $\Rightarrow a = b$ by Thm 1.2.6