Homework # 1 (Due Wednesday, January 22).

• Do the following problems:

1] Consider the following sets:

a) Set of all natural numbers $\mathbb{N} = \{1, 2, 3, 4, \dots\}$;

b) Set of all even integers $A = \{0, 2, -2, 4, -4, \dots\}$;

c) Set of all vectors on the plane;

d) Set of all polynomials of degree 2 with integer coefficients;

e) Set of all polynomials of degree ≤ 2 with integer coefficients;

f) Set of all rational numbers;

g) Set of all 2×2 matrices with rational entries.

For each set above decide if the set is a group under addition, a ring, a field, or neither. Please explain your reasoning, i.e. if some axiom for a group, ring, or field is violated, explain which one. (List of axioms is attached to this homework).

2] Read example 1.3 on p. 5 of the textbook.

Prove that if n is an integer such that n^2 is a multiple of 5, the n is also a multiple of 5.

Axioms for Real numbers.

Assume that a, b, c are arbitrary real numbers.

(A 1) (Closure under addition): $a \in \mathbb{R}$, $b \in \mathbb{R} \Rightarrow a+b \in \mathbb{R}$.

(A 2) (Existence of additive identity): $\exists\, 0 \in \mathbb{R}$: $a+0=0+a=a$

(A 3) (Existence of additive inverse): $a+(-a)=(-a)+a=0$

(A 4) (Associative law for addition): $(a+b)+c=a+(b+c)$.

(A 5) (Commutative law for addition): $a+b=b+a$

(A 6) (Closure under multiplication): $a \in \mathbb{R}$, $b \in \mathbb{R} \Rightarrow a \cdot b \in \mathbb{R}$.

(A 7) (Associative law under multiplication): $a \cdot (b+c) = a \cdot b + a \cdot c$

(A 8) (Existence of multiplicative identity): $\exists\, 1 \in \mathbb{R}$., $1 \cdot a = a \cdot 1 = a$.

(A 9) (Existence of multiplicative inverse): $a \cdot a^{-1} = a^{-1} \cdot a = 1$, for $a \neq 0$.

(A 10) (Commutative law for multiplication): $a \cdot b = b \cdot a$

(A 11) (Order axiom) every real number belongs to exactly one of three distinct sets: set of positive reals, set of negative reals, or set $\{0\}$.

- Axioms (A1)–(A4) make real numbers $\mathbb{R}$ a <u>group</u> under addition.
- Axioms (A1)–(A5) make $\mathbb{R}$ a <u>commutative (Abelian) group</u>.
- Axioms (A1)–(A8) make $\mathbb{R}$ a <u>ring</u>.
- Axioms (A1)–(A10) make $\mathbb{R}$ a <u>commutative field</u>.
- Axioms (A1)–(A11) make $\mathbb{R}$ a <u>commutative ordered field</u>.